QUANTUM $\mathrm{SL}(3,\mathbf{C})$'S WITH CLASSICAL REPRESENTATION THEORY

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ABSTRACT. We study and classify almost all quantum $\mathrm{SL}(3,\mathbf{C})$'s whose representation theory is "similar" to that of the (ordinary) group $\mathrm{SL}(3,\mathbf{C})$. Only one case, related to smooth elliptic curves, could not be treated completely.

1. Introduction

There are several approaches to the theory of quantum groups, depending on what aspect of group theory one wants to "quantize": one may consider the fundamental object to be, among other things, the algebra of continuous functions on a compact group [21], or the enveloping algebra of a complex semi-simple Lie algebra [8, 13], or the algebra of polynomial functions on a complex algebraic group [10]. Let us therefore begin by saying (definition 1.1 below) what we mean by a quantum analogue of (the algebra of polynomial functions on) a connected complex reductive group; roughly speaking, we ask the representation theory to be preserved. We do not claim that this is the only reasonable way to define "quantum reductive groups" (it is easy to see that it excludes, e.g., quantum tori), but it will be the one taken throughout the present paper.

So let G be a connected complex reductive group, B a Borel subgroup of G and P (resp. P^+) the set of integral (resp. dominant integral) weights of G w.r.t. B. For each $\lambda \in P^+$, denote by L_λ the simple G-module of highest weight λ and let $d_\lambda = \dim L_\lambda$. If $\lambda, \mu, \nu \in P^+$, let $m_{\lambda\mu\nu}$ be the multiplicity of L_ν in the decomposition of $L_\lambda \otimes L_\mu$.

Definition 1.1. We call a *quantum* G any (not necessarily commutative) Hopf algebra $\mathcal{A}$ (over $\mathbf{C}$) such that

(a) there is a family $\{V_\lambda \mid \lambda \in P^+\}$ of simple and pairwise nonisomorphic $\mathcal{A}$-comodules, with $\dim V_\lambda = d_\lambda$,
(b) every $\mathcal{A}$-comodule is isomorphic to a direct sum of these,
(c) for every $\lambda, \mu \in P^+$, $V_\lambda \otimes V_\mu$ is isomorphic to $\bigoplus_\nu m_{\lambda\mu\nu} V_\nu$.

Of course, the algebra $\mathcal{O}(G)$ of polynomial functions on the (ordinary) group G is a quantum G.

Recall that the tensor product of two $\mathcal{A}$-comodules is defined using the algebra structure on $\mathcal{A}$; thus, the idea is that the way a V_ν sits inside $V_\lambda \otimes V_\mu$ influences this algebra structure. In particular, the decomposition of $V_\lambda \otimes V_\lambda$ for a given quantum G may well be such that the exterior square $\wedge^2 V_\lambda$ is not a subcomodule; this basically accounts for the non-commutativity of such a quantum G.

There are (at least) two natural notions of equivalence for quantum G's, namely

- isomorphism of Hopf algebras,

- **C**-linear monoidal equivalence of categories of comodules (called *categorial* equivalence).

The second notion is weaker than the first. For example, the so-called "Jordanian" quantum SL(2) (introduced in [7]) is categorially equivalent to $\mathcal{O}(\mathrm{SL}(2))$ [22], but it is not commutative.

Up to categorial equivalence, quantum SL(n)'s have been classified in [14]: they are parametrized by a "deformation" parameter (either 1 or not a root of unity) and a "twisting" parameter (an n-th root of unity).

Up to Hopf algebra isomorphism, quantum SL(2)'s have been classified in [22].

In the present work, we study quantum SL(3)'s, up to Hopf algebra isomorphism. To read definition 1.1 in this case, recall that $P = \mathbf{Z}^2$, $P^+ = \mathbf{N}^2$, and that $d_{(k,\ell)} = (k+1)(\ell+1)(k+\ell+2)/2$ for $(k,\ell) \in \mathbf{N}^2$. Also, the multiplicities $m_{(k,\ell)\,(k',\ell')\,(k'',\ell'')}$ can be computed combinatorially (using, e.g., the Littlewood-Richardson rule).

If $\mathcal{A}$ is a quantum SL(3), the idea is to find data consisting of a finite number of $\mathcal{A}$-comodules and a finite number of $\mathcal{A}$-comodule morphisms between tensor products of them, such that $\mathcal{A}$ can be reconstructed (in the Tannaka-Krein sense) from these data, and to see that classifying quantum SL(3)'s up to isomorphism amounts to classifying these finite-dimensional data up to (a suitable notion of) equivalence.

In principle, these data could involve only the "natural" 3-dimensional comodule $V_{(1,0)}$, because definition 1.1(c) implies that every $V_{(k,\ell)}$ is contained in some tensor power of $V_{(1,0)}$, so its matrix coefficients generate $\mathcal{A}$ as an algebra. However, we rather use *both* "fundamental" comodules $V := V_{(1,0)}$ and $W := V_{(0,1)}$ (together with a suitable collection of morphisms): the point is that with these 18 generators (instead of 9), $\mathcal{A}$ can be presented by *quadratic* relations (instead of cubic ones, such as a "quantum determinant").

In section 3, we make these finite-dimensional data precise: starting from a given quantum SL(3), we choose eight morphisms between tensor products of its comodules V and W. The Schur lemma imposes some compatibility conditions between these morphisms. This leads us to the definition of a *basic quantum datum* (BQD for short), in which V and W become just vector spaces and the eight morphisms just linear maps, satisfying the compatibility conditions mentioned above.

Conversely, in section 4, we start from a BQD $\mathcal{L}$ and we reconstruct a Hopf algebra $\mathcal{A}_{\mathcal{L}}$ by the usual Tannaka-Krein procedure. The goal of sections 5–8 is to see whether this Hopf algebra is actually a quantum SL(3). In other words: if the fundamental comodules of a Hopf algebra are "SL(3)-ish", does it follow that *all* comodules are?

To understand sections 5 and 6, let us first recall the following well-known situation. Let G, B, P, P^+ be as at the beginning of this introduction, denote by U the unipotent radical of B and let T be a maximal torus in B. View elements of P as characters of T. Since T normalizes U, T acts from the right on G/U and from the left on $U\backslash G$; this induces P^+-gradings $\mathcal{O}(\overline{G/U}) = \bigoplus_{\lambda\in P^+} V_\lambda$ and $\mathcal{O}(\overline{U\backslash G}) = \bigoplus_{\lambda\in P^+} V^\lambda$. By the Borel-Weil theorem, the V_λ's are precisely the irreducible representations of G; therefore $\mathcal{O}(\overline{G/U})$ is called a *shape algebra* for G. Furthermore, V^λ can be identified with the dual of V_λ, so the algebra of T-invariants

$$\mathcal{G}(G) := \mathcal{O}(\overline{U\backslash G} \times \overline{G/U})^T = \bigoplus_{\lambda\in P^+} V^\lambda \otimes V_\lambda$$

identifies with $\mathcal{O}(G)$ as a vector space, by the Peter-Weyl decomposition. Actually, for a suitable additive function $h : P^+ \to \mathbf{N}$, $\mathcal{O}(G)$ becomes $\mathbf{N}$-filtered by putting $V^\lambda \otimes V_\lambda$ into degree $h(\lambda)$, and then $\mathcal{G}(G) \simeq \operatorname{gr} \mathcal{O}(G)$.

(Note that we have avoided using the opposite unipotent subgroup U^-. Also, the maximal torus T only appears in the guise of a P-grading and is not really used as a subgroup of G. This is necessary, because there exist quantum G's in which neither G/U^- nor T have quantum analogues: the Jordanian quantum SL(2) already mentioned is an easy example.)

In section 5, we define two $\mathbf{N}^2$-graded quadratic algebras $\mathcal{M}_\mathcal{L} = \bigoplus V_{(k,\ell)}$ and $\mathcal{N}_\mathcal{L} = \bigoplus V^{(k,\ell)}$ (generated by $V \oplus W$ and $V^* \oplus W^*$, respectively), which are quantum analogues of $\mathcal{O}(\overline{G/U})$ and $\mathcal{O}(\overline{U\backslash G})$ (for $G = \mathrm{SL}(3)$). We show that $\dim V_{(k,\ell)} = \dim V^{(k,\ell)} = d_{(k,\ell)}$ for all (k,ℓ) and that $\mathcal{M}_\mathcal{L}$, $\mathcal{N}_\mathcal{L}$ are Koszul algebras, except possibly when $\mathcal{L}$ is a so-called *elliptic* BQD (case I.h in the classification of section 10).

In section 6, we consider the subalgebra $\mathcal{G}_\mathcal{L}$ of $\mathcal{N}_\mathcal{L} \otimes \mathcal{M}_\mathcal{L}$ defined by

$$\mathcal{G}_\mathcal{L} := \bigoplus_{(k,\ell)\in\mathbf{N}^2} V^{(k,\ell)} \otimes V_{(k,\ell)}$$

(a quantum analogue of $\mathcal{G}(G)$), which is also $\mathbf{N}^2$-graded by putting $V^{(k,\ell)} \otimes V_{(k,\ell)}$ into degree (k,ℓ). We give a presentation of $\mathcal{G}_\mathcal{L}$ that can be deduced from a suitable presentation of $\mathcal{A}_\mathcal{L}$ by "cutting off" all terms of degree < 2; this yields a canonical surjection $\mathcal{G}_\mathcal{L} \to \operatorname{gr} \mathcal{A}_\mathcal{L}$. If $\mathcal{L}$ is not elliptic, we show (noting that $\mathcal{G}_\mathcal{L}$ is still Koszul) that this surjection is an isomorphism (a quantum analogue of $\mathcal{G}(G) \simeq \operatorname{gr} \mathcal{O}(G)$), using the results of [5]. Here lies the main advantage in dealing with algebras that are quadratic.

Section 7 is technical: we construct an endomorphism P of $V^{\otimes k} \otimes W^{\otimes \ell}$ (for each $(k,\ell) \in \mathbf{N}^2$) whose properties will be used in the next section.

In section 8, we finally show that if $\mathcal{L}$ is a nonelliptic BQD, then $\mathcal{A}_\mathcal{L}$ is indeed a quantum SL(3) (where the $V_{(k,\ell)}$ of definition 1.1 are those appearing in the $\mathbf{N}^2$-graded algebra $\mathcal{M}_\mathcal{L}$).

In section 9, we define an equivalence relation for BQD's and then summarize the previous results, showing that—away from the elliptic case—the correspondences $\mathcal{A} \mapsto \mathcal{L}_\mathcal{A}$ and $\mathcal{L} \mapsto \mathcal{A}_\mathcal{L}$ between quantum SL(3)'s (up to isomorphism) and BQD's (up to equivalence) are inverse of each other.

This raises the question of classifying BQD's. A related classification problem has been studied in [9], where necessary conditions are considered for a quantum analogue of $\mathcal{O}(\mathrm{GL}(3))$ to have correct dimensions in degrees ≤ 4. It turns out that these conditions, plus a quantum determinant being central, plus a parameter not being a root of unity, amount to our definition of a BQD.

Since we need an explicit classification of BQD's for a crucial case by case argument in section 5, we reproduce it here, in section 10. This classification is complete, except for case I.h, related to elliptic curves. (This case is however shown to exist.) By the results of section 9, this also yields a classification of all (nonelliptic) quantum SL(3)'s. An important ingredient in this classification will be a 3×3-matrix Q, which encodes the square of the antipode and which can take four different Jordan normal forms. The first possible form is the identity; we give a geometric description of some cases there, in terms of plane cubic curves. The second possible

form has three different eigenvalues; it leads in particular to the Artin-Schelter-Tate quantum SL(3)'s [2] (of which the standard quantum SL(3) [10] is a special case), and to the Cremmer-Gervais one (see [12]). The third and fourth forms are nondiagonal.

Finally, we list some indications for further study in section 11.

2. Notations and conventions

We denote by $\mathbf{Z}$ (resp. $\mathbf{N}$, $\mathbf{C}$) the set of integers (resp. nonnegative integers, complex numbers). If $n \in \mathbf{N}$, $n \geq 1$, and $t \in \mathbf{C}$, let

$$[n]_t := 1 + t + \cdots + t^{n-1}$$
$$[n]_t\,! := [1]_t\,[2]_t\,\ldots\,[n]_t$$

All vector spaces, algebras and tensor products are over $\mathbf{C}$. If X, Y are finite-dimensional vector spaces, we denote by $\mathrm{Lin}(X,Y)$ the space of linear maps from X to Y, and if $\alpha \in \mathrm{Lin}(X,Y)$, we denote by ${}^\tau\alpha \in \mathrm{Lin}(Y^*, X^*)$ its transpose. The identity map on X is denoted by 1_X (or simply 1). The tensor algebra of X is denoted by TX. The tensor product of X and Y will be denoted as usual by $X \otimes Y$, but for typographical reasons, we denote the tensor product of two linear maps α, β by (α, β).

If $\mathcal{A}$ is an algebra, we think of an element $\alpha \in \mathrm{Lin}(X,Y) \otimes \mathcal{A}$ as a "linear map with coefficients in $\mathcal{A}$", and we call *space of coefficients* of α the unique minimal vector subspace $\mathrm{Coeff}(\alpha)$ of $\mathcal{A}$ such that $\alpha \in \mathrm{Lin}(X,Y) \otimes \mathrm{Coeff}(\alpha)$; obviously, $\dim \mathrm{Coeff}(\alpha) \leq (\dim X)(\dim Y)$.

Note that an equality in $\mathrm{Lin}(X,Y) \otimes \mathcal{A}$ amounts to $(\dim X)(\dim Y)$ equalities in $\mathcal{A}$. We shall use this to write relations in $\mathcal{A}$ in a condensed way.

If $\alpha \in \mathrm{Lin}(X,Y) \otimes \mathcal{A}$ and $\beta \in \mathrm{Lin}(Y,Z) \otimes \mathcal{A}$, there is an obvious notion of composite $\beta\alpha \in \mathrm{Lin}(X,Z) \otimes \mathcal{A}$ (using the multiplication in $\mathcal{A}$). Similarly, if $\alpha \in \mathrm{Lin}(X,Y) \otimes \mathcal{A}$ and $\beta \in \mathrm{Lin}(X', Y') \otimes \mathcal{A}$, there is an obvious notion of tensor product $(\alpha, \beta) \in \mathrm{Lin}(X \otimes X', Y \otimes Y') \otimes \mathcal{A}$ (ditto).

All Hopf algebras are supposed to have an invertible antipode. "Comodule" means "finite-dimensional right comodule".

Let $\mathcal{A}$ be a Hopf algebra with comultiplication Δ, counit ε and antipode S. We view an $\mathcal{A}$-comodule structure on a finite-dimensional vector space X as an element $t \in \mathrm{Lin}(X,X) \otimes \mathcal{A}$ such that $\Delta(t) = t \otimes t$ and $\varepsilon(t) = 1_X$ (these are equalities in $\mathrm{Lin}(X,X) \otimes (\mathcal{A} \otimes \mathcal{A})$ and in $\mathrm{Lin}(X,X)$, respectively). Recall that every $\mathcal{A}$-comodule is a direct sum of simple ones if and only if $\mathcal{A}$ is the (direct) sum of the coefficient spaces of all (equivalence classes of) simple $\mathcal{A}$-comodules. If so, this direct sum is called the *Peter-Weyl decomposition* of $\mathcal{A}$.

An $\mathcal{A}$-comodule morphism (more simply called $\mathcal{A}$-morphism) between two $\mathcal{A}$-comodules (X,t) and (Y,u) is just an (ordinary) linear map $\alpha \in \mathrm{Lin}(X,Y)$ such that $\alpha t = u\alpha$, where composites are taken in the above sense. Tensor products of comodules also coincide with tensor products in the above sense. The left dual of a comodule (X,t) is $(X^*, {}^*t)$, where ${}^*t = S({}^\tau t) \in \mathrm{Lin}(X^*, X^*) \otimes \mathcal{A}$, and the right dual is (X^*, t^*), with $t^* = S^{-1}({}^\tau t)$. Recall that for the left (resp. right) dual structure, the canonical maps $X^* \otimes X \to \mathbf{C}$ and $\mathbf{C} \to X \otimes X^*$ (resp. $X \otimes X^* \to \mathbf{C}$ and $\mathbf{C} \to X^* \otimes X$) are $\mathcal{A}$-morphisms.

3. From quantum SL(3)'s to BQD's

Let $\mathcal{A}$ be a quantum SL(3) (see definition 1.1) and write $V := V_{(1,0)}$, $W := V_{(0,1)}$.

Proposition 3.1. *There are $\mathcal{A}$-morphisms*

$$\begin{array}{ll} A : V \otimes V \to W & a : W \to V \otimes V \\ B : W \otimes W \to V & b : V \to W \otimes W \\ C : W \otimes V \to \mathbf{C} & c : \mathbf{C} \to V \otimes W \\ D : V \otimes W \to \mathbf{C} & d : \mathbf{C} \to W \otimes V \end{array} \tag{3.1}$$

unique up to scalars, a constant $q \neq 0$, $q^2 \neq -1$, unique up to $q \leftrightarrow -q$ and $q \leftrightarrow q^{-1}$, and a unique 3-rd root of unity ω, such that

$$(1_V, C)(c, 1_V) = 1_V \qquad (D, 1_V)(1_V, d) = 1_V \tag{3.2a}$$
$$Aa = 1_W \tag{3.2b}$$
$$C(A, 1_V) = \omega\, D(1_V, A) \qquad (1_V, a)c = \omega\,(a, 1_V)d \tag{3.2c}$$
$$(1_V, D)(a, 1_W) = B \qquad \omega^2\,(1_W, A)(d, 1_V) = b \tag{3.2d}$$
$$\omega\,(C, 1_V)(1_W, a) = B \qquad (A, 1_W)(1_V, c) = b \tag{3.2e}$$
$$Dc = \kappa\, 1_{\mathbf{C}} \qquad Cd = \kappa\, 1_{\mathbf{C}} \tag{3.2f}$$

$$(1_V, A)(a, 1_V)(A, 1_V)(1_V, a) = \rho\,(1_{V\otimes W} + cD) \tag{3.2g}$$
$$(A, 1_V)(1_V, a)(1_V, A)(a_V, 1) = \rho\,(1_{W\otimes V} + dC) \tag{3.2h}$$

where $\kappa = q^{-2} + 1 + q^2$ and $\rho = (q + q^{-1})^{-2}$.

(Recall that we write (α, β) for the tensor product of any two linear maps α, β.) Note that q^2 and ω are the two parameters of [14] mentioned in the introduction.

Proof. First, definition 1.1(c) asks for the following decompositions:

$$\begin{array}{ll} V \otimes V \simeq W \oplus V_{(2,0)} & V \otimes W \simeq \mathbf{C} \oplus V_{(1,1)} \\ W \otimes V \simeq \mathbf{C} \oplus V_{(1,1)} & W \otimes W \simeq V \oplus V_{(0,2)} \end{array} \tag{3.3}$$

This already implies the desired uniqueness of the maps (3.1).

Furthermore, (3.3) prevents V from being its own dual; so the (left *and* right) dual of V must be W, and vice-versa. Hence there exist $\mathcal{A}$-morphisms C, c, D, d as in (3.1) satisfying (3.2a).

Again by (3.3), there are $\mathcal{A}$-morphisms A, a as in (3.1) satisfying (3.2b). By definition 1.1(c), $V^{\otimes 3}$ contains exactly one copy of the trivial $\mathcal{A}$-comodule $\mathbf{C}$, hence

$$\begin{array}{c} C(A, 1_V) = \lambda\, D(1_V, A) \\ (1_V, a)c = \mu\,(a, 1_V)d \end{array} \qquad (\lambda, \mu \in \mathbf{C})$$

We then define

$$B := (1_V, D)(a, 1_W) = \mu\,(C, 1_V)(1_W, a)$$
$$b := \lambda\mu\,(1_W, A)(d, 1_V) = \lambda^2\mu\,(A, 1_W)(1_V, c)$$

As a consequence

$$\begin{aligned} Bb &= \lambda\mu\,(1_V, D)(a, 1_W)(1_W, A)(d, 1_V) = \lambda\mu\,(1_V, D)(1_V, 1_V, A)(a, 1_V, 1_V)(d, 1_V) \\ &= (C, 1_V)(1_V, A, 1_V)(1_V, a, 1_V)(1_V, c) = (C, 1_V)(1_V, c) = 1_V \\ Bb &= \lambda^2\mu^2(C, 1_V)(1_W, a)(A, 1_W)(1_V, c) = \lambda^2\mu^2(C, 1_V)(A, 1_V, 1_V)(1_V, 1_V, a)(1_V, c) \\ &= \lambda^3\mu^3(1_V, D)(1_V, A, 1_V)(1_V, a, 1_V)(d, 1_V) = \lambda^3\mu^3(1_V, D)(d, 1_V) = \lambda^3\mu^3\,1_V \end{aligned}$$

so there is a 3-rd root of unity ω such that $\lambda\mu = \omega^2$. Moreover, any rescaling of the maps (3.1) that leaves the relations (3.2ab) intact also leaves $\lambda\mu$ invariant, hence ω is unique. Actually, there is such a rescaling after which $\lambda = \mu = \omega$. This yields (3.2cde). Now

$$Cd = C(A, 1_V)(a, 1_V)d = D(1_V, A)(1_V, a)c = Dc$$

which implies (3.2f). Next, define

$$\begin{aligned} F &:= (A, 1_V)(1_V, a) : V \otimes W \to W \otimes V \\ G &:= (1_V, A)(a, 1_V) : W \otimes V \to V \otimes W \end{aligned} \tag{3.4}$$

In view of (3.3), we must have

$$\begin{aligned} GF &= (1_V, A)(a, 1_V)(A, 1_V)(1_V, a) = \rho\,1_{V\otimes W} + \sigma\,cD \\ FG &= (A, 1_V)(1_V, a)(1_V, A)(a_V, 1) = \rho'\,1_{W\otimes V} + \sigma'\,dC \end{aligned} \tag{3.5}$$

for some $\rho, \sigma, \rho', \sigma' \in \mathbf{C}$, which are unique because any rescaling that leaves (3.2abc) intact also leaves (3.5) intact. Now use

$$Fc = \omega\,d \qquad CF = \omega\,D \qquad Gd = \omega^2\,c \qquad DG = \omega^2\,C$$

to compare $(GF)(GF) = G(FG)F$ and $(FG)(FG) = F(GF)G$: we get $\rho'(\rho - \rho') = 0$ and $\rho(\rho' - \rho) = 0$, so $\rho' = \rho$. Multiplying (3.5) on the right by c (resp. d) then yields $1 = \rho + \kappa\sigma$ and $1 = \rho + \kappa\sigma'$, so $\sigma' = \sigma$. Next,

$$\begin{aligned} B(A, 1_W)(1_V, G) &= \omega^2\rho(D, 1_V) + \omega^2\sigma(1_V, C) \\ = B(1_W, A)(F, 1_V) &= \omega^2\rho(1_V, C) + \omega^2\sigma(D, 1_V) \end{aligned}$$

so $\sigma = \rho$, which gives (3.2gh). Since $\rho \neq 0$, the condition $\rho = (q + q^{-1})^{-2}$ defines q with the desired uniqueness, and $\kappa = \rho^{-1} - 1 = q^{-2} + 1 + q^2$. □

Proposition 3.2. *In the notations of proposition 3.1, either $q^2 = 1$, or q^2 is not a root of unity.*

Proof. Let $R := q\,1_{V\otimes V} - (q + q^{-1})\,aA$. It follows from (3.2) that

$$(R, 1_V)(1_V, R)(R, 1_V) = (1_V, R)(R, 1_V)(1_V, R) \tag{3.6a}$$

$$(R - q)(R + q^{-1}) = 0 \tag{3.6b}$$

If $k \geq 2$, define the following endomorphisms of $V^{\otimes k}$:

$$R_i := 1_{V^{\otimes(k-i-1)}} \otimes R \otimes 1_{V^{\otimes(i-1)}} \qquad (1 \leq i \leq k - 1)$$

(This unusual right-to-left numbering will be convenient in section 7.)

Next, let us recall some general folklore on Hecke calculus. Let Sym_k be the symmetric group on $\{1, \ldots, k\}$ and denote the transposition $(i, i+1)$ by s_i ($1 \leq i \leq k-1$). If $w \in \mathrm{Sym}_k$ and if $w = s_{i_1} \ldots s_{i_p}$ is an expression of minimal length

$p =: \ell(w)$, let $R_w = R_{i_1} \dots R_{i_p}$; it follows from (3.6a) that R_w does not depend on the choice of a particular such expression. Now let

$$S_k := \sum_{w \in \mathrm{Sym}_k} q^{\ell(w)} R_w \tag{3.7}$$

It is easy to see that

$$R_i S_k = S_k R_i = q\, S_k \tag{3.8}$$

for every i (use the fact that $\ell(s_i w) = \ell(w s_i) = \ell(w) \pm 1$ for all w and split the sum defining S_k into two sums accordingly; then use (3.6b)).

Recall also that every $w \in \mathrm{Sym}_k$ has a unique expression of minimal length of the form $w = v_1 v_2 \dots v_{k-1}$, where

$$\begin{aligned} v_1 &\in \{\, 1\,,\, s_1 \,\} \\ v_2 &\in \{\, 1\,,\, s_2\,,\, s_2 s_1 \,\} \\ &\vdots \\ v_{k-1} &\in \{1\,,\, s_{k-1}, s_{k-1}s_{k-2}\,,\, \dots\,,\, s_{k-1}s_{k-2}\dots s_1 \,\} \end{aligned}$$

(This is just the bubble-sort principle.) It follows that

$$\begin{aligned} S_k = (1 + qR_1)(1 + qR_2 + q^2 R_2 R_1) \times \cdots \\ \cdots \times (1 + qR_{k-1} + \cdots + q^{k-1} R_{k-1} \dots R_1) \end{aligned} \tag{3.9}$$

In particular, S_k is of the form

$$S_k = [k]_{q^2}\,! + a_{k-1} * + \cdots + a_1 * \tag{3.10}$$

(where a_i is defined the same way as R_i).

Now assume that q^2 is a primitive n-th root of unity, $n \geq 2$. From proposition 3.1, we already know that $n \geq 3$.

Claim A. *If $2 \leq k \leq n$, then $S_k \neq 0$.*

First case: $k < n$. As an $\mathcal{A}$-comodule, $\mathrm{Im}\, a_i$ is isomorphic to $V^{\otimes(k-i-1)} \otimes W \otimes V^{\otimes(i-1)}$, so by definition 1.1(c), $\mathrm{Im}\, a_i$ does not contain $V_{(k,0)}$. It follows that $\sum_{i=1}^{k-1} (\mathrm{Im}\, a_i)$ is a proper submodule of $V^{\otimes k}$. Now apply (3.10), noting that $[k]_{q^2}\,! \neq 0$.

Second case: $k = n$. (Adapted from [14, section 4].) It follows from (3.2) that

$$(1_{V^{\otimes(n-1)}}, D)(R_{n-1}, 1_W)(1_{V^{\otimes(n-1)}}, c) = q^3\, 1_{V^{\otimes(n-1)}}$$

Using this, (3.9) and (3.8), it follows that

$$\begin{aligned} (1_{V^{\otimes(n-1)}}, D)(S_n, 1_W)(1_{V^{\otimes(n-1)}}, c) &= (q^{-2} + 1 + q^2 + q^4 + \cdots + q^{2n})\, S_{n-1} \\ &= (q^{-2} + 1)\, S_{n-1} \end{aligned}$$

Since $S_{n-1} \neq 0$ and $q^2 \neq -1$, this shows claim A.

By definition 1.1(c), $V^{\otimes k}$ contains exactly one copy of $V_{(k,0)}$; let us denote it by $M_{(k)}$.

Claim B. *For $k \leq n$,* $\mathrm{Im}\, S_k = M_{(k)}$.

(Proof adapted from [14, lemma 4.4].) We proceed by induction over k. If $k = 2$, the statement is clear from the definitions, so assume $2 < k \leq n$. Let $S'_k := (S_{k-1}, 1_V)$ and $S''_k := (1_V, S_{k-1})$. By (3.9), $S_k = S'_k T'_k$ for some T'_k, and similarly, $S_k = S''_k T''_k$, so $\mathrm{Im}\, S_k \subset \mathrm{Im}\, S'_k \cap \mathrm{Im}\, S''_k$. By induction, $\mathrm{Im}\, S'_k = M_{(k-1)} \otimes V$ and $\mathrm{Im}\, S''_k = V \otimes M_{(k-1)}$. By definition 1.1(c), both images are isomorphic to the

comodule $V_{(k,0)} \oplus V_{(k-2,1)}$, so since $V^{\otimes k}$ contains only one copy of $V_{(k,0)}$ (namely $M_{(k)}$), either $\operatorname{Im} S'_k \cap \operatorname{Im} S''_k = M_{(k)}$, or $\operatorname{Im} S'_k = \operatorname{Im} S''_k$. But the second possibility would imply the following equalities of subspaces in $V^{\otimes(2k)}$:

$$M_{(k)} \otimes V^{\otimes k} = V \otimes M_{(k)} \otimes V^{\otimes(k-1)} = \cdots = V^{\otimes k} \otimes M_{(k)}$$

which is absurd. Therefore, $\operatorname{Im} S_k \subset M_{(k)}$, so by claim A and by the simplicity of $M_{(k)}$, this shows claim B.

Now let $N_{(k)}$ be the unique supplemental comodule of $M_{(k)}$ in $V^{\otimes k}$. From the proof of claim A, it follows that $\operatorname{Im}(S_k - [k]_{q^2}!) \subset N_{(k)}$ for all k. But for $k = n$, this contradicts claim B, because $[n]_{q^2}! = 0$. □

Definition 3.3. A *basic quantum* SL(3) *datum* (BQD for short) consists of two 3-dimensional vector spaces V and W, together with eight linear maps (3.1) satisfying (3.2), and such that either $q^2 = 1$, or q^2 is not a root of unity.

If $\mathcal{A}$ is a quantum SL(3), we denote by $\mathcal{L}_{\mathcal{A}}$ the associated BQD.

A straightforward computation shows that in a BQD, relations (3.2) with $V \leftrightarrow W$, $A \leftrightarrow B$, $a \leftrightarrow b$, $C \leftrightarrow D$, $c \leftrightarrow d$ interchanged are also satisfied, i.e.,

$$\begin{aligned}
&(1_W, D)(d, 1_W) = 1_W && (C, 1_W)(1_W, c) = 1_W \\
&Bb = 1_V && \\
&D(B, 1_W) = \omega\, C(1_W, B) && (1_W, b)d = \omega\, (b, 1_W)c \\
&(1_W, C)(b, 1_V) = A && \omega^2\, (1_V, B)(c, 1_W) = a \\
&\omega\, (D, 1_W)(1_V, b) = A && (B, 1_V)(1_W, d) = a \\
&(1_W, B)(b, 1_W)(B, 1_W)(1_W, b) = \rho\, (1_{W\otimes V} + dC) && \\
&(B, 1_W)(1_W, b)(1_W, B)(b, 1_W) = \rho\, (1_{V\otimes W} + cD) &&
\end{aligned} \tag{3.11}$$

If $t \in \operatorname{Lin}(V, V) \otimes \mathcal{A}$ and $u \in \operatorname{Lin}(W, W) \otimes \mathcal{A}$ denote the $\mathcal{A}$-comodule structures on V and W, it follows from definition 1.1(c) that the coefficients of t and u generate $\mathcal{A}$. Moreover, the $\mathcal{A}$-morphisms (3.1) induce the following $4 \times 27 + 4 \times 9$ relations in $\mathcal{A}$:

$$\begin{matrix}
A(t,t) = uA & B(u,u) = tB & C(u,t) = C & D(t,u) = D \\
(t,t)a = au & (u,u)b = bt & (t,u)c = c & (u,t)d = d
\end{matrix} \tag{3.12}$$

(In coordinates, this reads $A^{\alpha}_{ij} t^i_k t^j_\ell = u^\alpha_\beta A^\beta_{k\ell}$, etc.)

4. From BQD's to Hopf algebras

Let us now work the other way round: if $\mathcal{L}$ is a BQD, the usual Tannakian reconstruction procedure associates to it a bialgebra $\mathcal{A}_{\mathcal{L}}$, uniquely up to unique isomorphism, together with $\mathcal{A}_{\mathcal{L}}$-comodule structures $t \in \operatorname{Lin}(V, V) \otimes \mathcal{A}_{\mathcal{L}}$ and $u \in \operatorname{Lin}(W, W) \otimes \mathcal{A}_{\mathcal{L}}$, satisfying the two following properties:

$$A, a, B, b, C, c, D, d \text{ are } \mathcal{A}_{\mathcal{L}}\text{-morphisms} \tag{4.1a}$$

$$(\mathcal{A}, t, u) \text{ is universal with respect to (4.1a)} \tag{4.1b}$$

Condition (4.1b) means that if $(\mathcal{A}', t', u')$ also satisfies (4.1a), then there is a unique bialgebra homomorphism $\varphi : \mathcal{A}_{\mathcal{L}} \to \mathcal{A}'$ such that $\varphi(t) = t'$ and $\varphi(u) = u'$.

Explicitely, $\mathcal{A}_{\mathcal{L}}$ is generated by the $(9+9)$-dimensional space $\operatorname{Coeff}(t) + \operatorname{Coeff}(u)$, and relations (3.12) form a presentation of $\mathcal{A}_{\mathcal{L}}$.

The assignments $\Delta(t) = t \otimes t$, $\Delta(u) = u \otimes u$, $\varepsilon(t) = 1_V$, $\varepsilon(u) = 1_W$ can be uniquely extended to algebra homomorphisms $\Delta : \mathcal{A}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}} \otimes \mathcal{A}_{\mathcal{L}}$ and $\varepsilon : \mathcal{A}_{\mathcal{L}} \to \mathbf{C}$, which turn $\mathcal{A}_{\mathcal{L}}$ into a bialgebra.

Moreover, relations (3.2a) turn V and W into each other's left dual (in the monoidal category of $\mathcal{A}_{\mathcal{L}}$-comodules). This suggests an antipode on $\mathcal{A}_{\mathcal{L}}$ defined by

$$S(t) = c^\flat \, {}^\tau u \, C^\sharp \qquad S(u) = d^\flat \, {}^\tau t \, D^\sharp \tag{4.2}$$

(where $c^\flat$ is just c viewed as a linear map $W^* \to V$, and similarly for $C^\sharp : V \to W^*$, $d^\flat : V^* \to W$, $D^\sharp : W \to V^*$). Indeed, using (3.2) and (3.11), one checks that (4.2) uniquely extends to an algebra antihomomorphism $S : \mathcal{A}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$, and this turns $\mathcal{A}_{\mathcal{L}}$ into a Hopf algebra.

5. The shape algebra of a BQD

To a BQD $\mathcal{L}$, we associate the *shape algebra* $\mathcal{M}_{\mathcal{L}}$, defined as follows: let I^G be the ideal in $T(V \oplus W)$ generated by all elements $w \otimes v + (q + q^{-1})\, G(w \otimes v)$, where $v \in V$, $w \in W$ (the map G is defined in (3.4)). Then define $\mathcal{M}_{\mathcal{L}} := T(V \oplus W)/I$, where I is the ideal generated by $\operatorname{Im} a$, $\operatorname{Im} b$, $\operatorname{Im} c$ and I^G. (Note that I also contains $\operatorname{Im} d$ and all elements $v \otimes w + (q + q^{-1})\, F(v \otimes w)$.)

The natural $\mathbf{N}^2$-grading on $\mathrm{T}(V \oplus W)$, with V living in degree $(1,0)$ and W in degree $(0,1)$, factors to an $\mathbf{N}^2$-grading $\mathcal{M}_{\mathcal{L}} = \bigoplus V_{(k,\ell)}$. There are natural identifications $V_{(0,0)} \simeq \mathbf{C}$, $V_{(1,0)} \simeq V$, $V_{(0,1)} \simeq W$, $V_{(2,0)} \simeq \operatorname{Ker} A$, $V_{(1,1)} \simeq \operatorname{Ker} C$, $V_{(1,1)} \simeq \operatorname{Ker} D$, $V_{(0,2)} \simeq \operatorname{Ker} B$.

Lemma 5.1. *We have $\mathcal{M}_{\mathcal{L}} \simeq (TV \otimes TW)/I'$ (as $\mathbf{N}^2$-graded vector spaces), where*

$$I' = TV \otimes \operatorname{Im} a \otimes TV \otimes TW + TV \otimes \operatorname{Im} c \otimes TW + TV \otimes TW \otimes \operatorname{Im} b \otimes TW$$

Proof. First, the relations in I^G can be turned into a reduction system (in the sense of [4]), which has no ambiguities at all; so by the diamond lemma [4, thm 1.2],

$$T(V \oplus W) = (TV \otimes TW) \oplus I^G \tag{5.1}$$

Next, if $w, w' \in W$, it follows from (3.2) that

$$(q + q^{-1})^2\, (1, G)(G, 1)\big(w \otimes a(w')\big) = a(w) \otimes w' + B(w \otimes w') \otimes c(1)$$

This implies that

$$\begin{aligned} w \otimes a(w') = {} & a(w) \otimes w' + B(w \otimes w') \otimes c(1) \\ & - (q + q^{-1}) \Big((q + q^{-1})\, (1, G)(G, 1)\big(w \otimes a(w')\big) + (G, 1)\big(w \otimes a(w')\big) \Big) \\ & + \Big((q + q^{-1})\, (G, 1)\big(w \otimes a(w')\big) + w \otimes a(w') \Big) \end{aligned}$$

Therefore

$$W \otimes \operatorname{Im} a \subset \operatorname{Im} a \otimes W + V \otimes \operatorname{Im} c + I^G$$

Similarly,

$$\begin{aligned} \operatorname{Im} b \otimes V &\subset V \otimes \operatorname{Im} b + \operatorname{Im} d \otimes W + I^G \\ \operatorname{Im} c \otimes V &\subset \operatorname{Im} a \otimes V + I^G \\ \operatorname{Im} d &\subset \operatorname{Im} c + I^G \end{aligned}$$

Applied inductively, these rules show that $I = I' + I^G$, hence $I \cap (TV \otimes TW) = I'$ by (5.1). The result follows. $\square$

(There is of course a similar identification $\mathcal{M}_{\mathcal{L}} \simeq (TW \otimes TV)/I''$.)

Lemma 5.2. *We have* $\dim I_{(3,0)} = \dim I_{(0,3)} = 17$ *and* $\dim I_{(2,1)} = \dim I_{(1,2)} = 66$.

Proof. (We only look at $I_{(3,0)}$ and $I_{(2,1)}$.) Consider an element of $\mathrm{Im}\, a \otimes V \cap V \otimes \mathrm{Im}\, a$, say $(a,1)(x) = (1,a)(y)$, where $x \in W \otimes V$ and $y \in V \otimes W$. Applying $(A,1)$ and $(1,A)$ to this equality yields $x = F(y)$ and $G(x) = y$, respectively, so $y = GF(y) = \rho\, y + \rho\, c D(y)$. But $q^4 + q^2 + 1 \neq 0$ implies $\rho \neq 1$, hence $(1,a)(y) \in \mathrm{Im}(1,a)c$. Conversely (see (3.2c)), $\mathrm{Im}(1,a)c = \mathrm{Im}(a,1)d \subset \mathrm{Im}\, a \otimes V \cap V \otimes \mathrm{Im}\, a$. It follows that $I_{(3,0)} = \mathrm{Im}\, a \otimes V + V \otimes \mathrm{Im}\, a$ is of dimension $9 + 9 - 1 = 17$.

Similarly, consider an element of $\mathrm{Im}\, a \otimes W \cap V \otimes \mathrm{Im}\, c$, say $(a,1)(x) = (1,c)(y)$, where $x \in W \otimes W$ and $y \in V$. Applying $(A,1)$ and $(1,D)$ yields $x = b(y)$ and $B(x) = \kappa\, y$, so $y = Bb(y) = \kappa\, y$. But $q^4 \neq -1$ implies $\kappa \neq 1$, so $y = 0$. Thus, $I'_{(2,1)} = \mathrm{Im}\, a \otimes W + V \otimes \mathrm{Im}\, c$ is of dimension $9+3-0 = 12$. Now apply lemma 5.1. $\square$

Proposition 5.3. *If $\mathcal{L}$ is not elliptic (i.e., not case I.h in section 10), then there are bases (x_1, x_2, x_3) of V and (y_1, y_2, y_3) of W such that the monomials $x_1^a x_2^b x_3^c y_2^\ell y_1^k$ and $x_1^a x_2^b y_3^m y_2^\ell y_1^k$ $(a, b, c, k, \ell, m \in \mathbf{N})$ form a basis of $\mathcal{M}_{\mathcal{L}}$. Moreover, $\mathcal{M}_{\mathcal{L}}$ is a Koszul algebra.*

Proof. Using the classification given in section 10, one can check case by case that the relations coming from $\mathrm{Im}\, a$ may be written in the form

$$\begin{aligned} x_3 x_2 &= \text{terms in } x_1^2,\, x_1x_2,\, x_1x_3,\, x_2x_1,\, x_2^2,\, x_2x_3,\, x_3x_1 \\ x_3 x_1 &= \text{terms in } x_1^2,\, x_1x_2,\, x_1x_3,\, x_2x_1,\, x_2^2 \\ x_2 x_1 &= \text{terms in } x_1^2,\, x_1x_2 \end{aligned}$$

those coming from $\mathrm{Im}\, b$ in the form

$$\begin{aligned} y_2 y_3 &= \text{terms in } y_3^2,\, y_3y_2 \\ y_1 y_3 &= \text{terms in } y_3^2,\, y_3y_2,\, y_3y_1,\, y_2y_3,\, y_2^2 \\ y_1 y_2 &= \text{terms in } y_3^2,\, y_3y_2,\, y_3y_1,\, y_2y_3,\, y_2^2,\, y_2y_1,\, y_1y_3 \end{aligned}$$

that coming from $\mathrm{Im}\, c$ in the form

$$x_3 y_3 = \text{terms in } x_1y_1,\, x_1y_2,\, x_1y_3,\, x_2y_2,\, x_2y_3$$

and those coming from I^G in the form

$$y_\alpha x_i = \text{terms in } x_j y_\beta \ (j = 1,2,3,\ \beta = 1,2,3)$$

Now order the generators as follows: $x_1 < x_2 < x_3 < y_3 < y_2 < y_1$; then order the monomials degree-lexicographically. The relations defining $\mathcal{M}_{\mathcal{L}}$, as written above, take the form of a reduction system (in the sense of [4]) that is compatible with the ordering just defined (i.e., each term in a R.H.S. is strictly smaller than the corresponding L.H.S.). Since the relations are homogeneous of degree 2, all ambiguities live in degree 3, and since there are 50 irreducible monomials of degree 3, we know *in advance* that all ambiguities are resolvable, thanks to lemma 5.2. (This saves us from dozens of pages of ambiguity cumputations!) Thus, the first statement follows from the diamond lemma [4, thm 1.2], noting that the irreducible monomials are exactly those in the statement.

Finally, the basis so obtained is a labeled basis in the sense of [19], hence the last statement follows from [19, thm 5.3]. $\square$

The problem with the elliptic case (I.h in section 10) is that already the first three relations, viz.

$$\alpha x_2x_3 + \beta x_3x_2 + \gamma x_1^2 = 0$$
$$\alpha x_3x_1 + \beta x_1x_3 + \gamma x_2^2 = 0$$
$$\alpha x_1x_2 + \beta x_2x_1 + \gamma x_3^2 = 0$$

do not seem to be compatible with any semigroup ordering on the monomials, so the diamond lemma does not apply.

Corollary 5.4. *If $\mathcal{L}$ is not elliptic,* $\dim V_{(k,\ell)} = d_{(k,\ell)} = (k+1)(\ell+1)(k+\ell+2)/2$.

Similarly, define the *dual shape algebra* $\mathcal{N}_{\mathcal{L}} := \mathrm{T}(V^* \oplus W^*)/J$, where J is the ideal generated by $\operatorname{Im} {}^{\tau}\!A$, $\operatorname{Im} {}^{\tau}\!B$, $\operatorname{Im} {}^{\tau}\!D$ and J^F, the latter being the ideal generated by all elements $\eta \otimes \xi + (q + q^{-1})\ {}^{\tau}\!F(\eta \otimes \xi)$ ($\xi \in V^*$, $\eta \in W^*$).

Again, we have an obvious $\mathbf{N}^2$-grading $\mathcal{N}_{\mathcal{L}} = \bigoplus V^{(k,\ell)}$, and the results of this section hold for $\mathcal{N}_{\mathcal{L}}$ just as they do for $\mathcal{M}_{\mathcal{L}}$.

Question. When $\mathcal{L}$ is elliptic, is it still true that $\dim V_{(k,\ell)} = \dim V^{(k,\ell)} = d_{(k,\ell)}$ and that $\mathcal{M}_{\mathcal{L}}$, $\mathcal{N}_{\mathcal{L}}$ are Koszul?

6. Filtration of $\mathcal{A}_{\mathcal{L}}$ and dimensions

Consider the following subalgebra of $\mathcal{N}_{\mathcal{L}} \otimes \mathcal{M}_{\mathcal{L}}$:

$$\mathcal{G}_{\mathcal{L}} := \bigoplus_{(k,\ell)\in\mathbf{N}^2} V^{(k,\ell)} \otimes V_{(k,\ell)}$$

It is $\mathbf{N}^2$-graded (or $\mathbf{N}$-graded) by putting $V^{(k,\ell)} \otimes V_{(k,\ell)}$ into degree (k,ℓ) (or $k+\ell$).

Proposition 6.1. *The algebra $\mathcal{G}_{\mathcal{L}}$ is generated by $(V^* \otimes V) \oplus (W^* \otimes W)$, and presented by the relations*

$$\begin{aligned} &A(t,t) = 0 \qquad B(u,u) = 0 \qquad D(t,u) = 0 \\ &(t,t)a = 0 \qquad (u,u)b = 0 \qquad (t,u)c = 0 \\ &\qquad (u,t)F - F(t,u) = 0 \end{aligned} \tag{6.1}$$

(Here, t denotes the canonical $(V^* \otimes V)$-comodule structure on V, i.e., in coordinates, $t^i_j = x^i \otimes x_j$; similarly for u w.r.t. W.)

Proof. Let K be the ideal of $T\big((V^* \otimes V) \oplus (W^* \otimes W)\big)$ generated by the L.H.S.'s of (6.1): it follows from the definition of $\mathcal{M}_{\mathcal{L}}$ and of $\mathcal{N}_{\mathcal{L}}$ that we have an $\mathbf{N}^2$-graded map

$$T\big((V^* \otimes V) \oplus (W^* \otimes W)\big)/K \to \mathcal{G}_{\mathcal{L}} \tag{6.2}$$

It already follows from lemma 5.1 that this map is surjective. Since F is invertible, we have

$$T\big((V^* \otimes V) \oplus (W^* \otimes W)\big) = T(V^* \otimes V) \otimes T(W^* \otimes W) + K$$

so for each $(k,\ell) \in \mathbf{N}^2$, (6.2) restricts to a surjection

$$\begin{aligned} &V^{*\otimes k} \otimes V^{\otimes k} \otimes W^{*\otimes \ell} \otimes W^{\otimes \ell} \Big/ K \cap \big(V^{*\otimes k} \otimes V^{\otimes k} \otimes W^{*\otimes \ell} \otimes W^{\otimes \ell}\big) \\ &\to V^{*\otimes k} \otimes W^{*\otimes \ell} \otimes V^{\otimes k} \otimes W^{\otimes \ell} \Big/ \big(V^{*\otimes k} \otimes W^{*\otimes \ell} \otimes I'_{(k,\ell)} + J'_{(k,\ell)} \otimes V^{\otimes k} \otimes W^{\otimes \ell}\big) \end{aligned}$$

(where J' is to $\mathcal{N}_{\mathcal{L}}$ what I' is to $\mathcal{M}_{\mathcal{L}}$). Actually, this surjection is just the obvious map induced from the flip $V^{\otimes k} \otimes W^{*\otimes \ell} \to W^{*\otimes \ell} \otimes V^{\otimes k}$, so in view of the definitions of I', J' and K, it is also injective. $\square$

Note that (6.1) is just the "homogeneous part" of the following presentation of $\mathcal{A}_{\mathcal{L}}$:

$$\begin{gathered} A(t,t) = uA \qquad B(u,u) = tB \qquad D(t,u) = D \\ (t,t)a = au \qquad (u,u)b = bt \qquad (t,u)c = c \\ (u,t)F - F(t,u) = 0 \end{gathered} \tag{6.3}$$

so there is a canonical $\mathbf{N}$-graded surjection $\mathcal{G}_{\mathcal{L}} \to \operatorname{gr} \mathcal{A}_{\mathcal{L}}$, where we filter $\mathcal{A}_{\mathcal{L}} = \bigcup_{n \geq 0} \mathcal{A}^{(n)}$ by putting $\operatorname{Coeff}(t) + \operatorname{Coeff}(u)$ into degree 1.

Proposition 6.2. *If $\mathcal{L}$ is not elliptic, the surjection $\mathcal{G}_{\mathcal{L}} \to \operatorname{gr} \mathcal{A}_{\mathcal{L}}$ is an isomorphism.*

Proof. Let $X := (V^* \otimes V) \oplus (W^* \otimes W)$ and let K_2 be the subspace of $X \otimes X$ generated by the L.H.S.'s of (6.3), so $\mathcal{G}_{\mathcal{L}} = \mathrm{T}(X)/(K_2)$. Consider the map $\lambda : K_2 \to X$ sending each L.H.S. of (6.3) to the degree 1 part of its R.H.S., and similarly $\mu : K_2 \to \mathbf{C}$, for the degree 0 part; they are easily seen to be well-defined. Thanks to [5, thm 0.5 and lemma 3.3], we only have to prove the following four conditions:

(a) $\mathcal{G}_{\mathcal{L}}$ is Koszul,
(b) the image of $\lambda \otimes 1_X - 1_X \otimes \lambda$ (defined on $K_2 \otimes X \cap X \otimes K_2$) lies in K_2,
(c) $\lambda(\lambda \otimes 1_X - 1_X \otimes \lambda) = -(\mu \otimes 1_X - 1_X \otimes \mu)$,
(d) $\mu(\lambda \otimes 1_X - 1_X \otimes \lambda) = 0$.

To prove condition (a), let us temporarily change signs in the grading of $\mathcal{N}_{\mathcal{L}}$, i.e., put $V^{(k,\ell)}$ into degree $(-k,-\ell)$. For the resulting total $\mathbf{Z}^2$-grading on $\mathcal{N}_{\mathcal{L}} \otimes \mathcal{M}_{\mathcal{L}}$, we now have $\mathcal{G}_{\mathcal{L}} = (\mathcal{N}_{\mathcal{L}} \otimes \mathcal{M}_{\mathcal{L}})_{(0,0)}$. Since $\mathcal{M}_{\mathcal{L}}$ and $\mathcal{N}_{\mathcal{L}}$ are Koszul (proposition 5.3), $\mathcal{N}_{\mathcal{L}} \otimes \mathcal{M}_{\mathcal{L}}$ is also Koszul by [19, prop. 2.1], and since the quadratic relations defining $\mathcal{N}_{\mathcal{L}} \otimes \mathcal{M}_{\mathcal{L}}$ are homogeneous w.r.t. our temporary $\mathbf{Z}^2$-grading, $\mathcal{G}_{\mathcal{L}}$ is still Koszul.

We now abandon this temporary grading and consider $\mathcal{G}_{\mathcal{L}}$ as an $\mathbf{N}^2$-graded algebra, as before. Conditions (b), (c), (d) may clearly be checked separately in degrees $(3,0)$, $(2,1)$, $(1,2)$, $(0,3)$. We only look at degrees $(3,0)$ and $(2,1)$, since the other two are similar.

To improve legibility, we introduce the following notation: if L is a linear map, we write L_i for a tensor product $(1,\ldots,1,L,1,\ldots,1)$, with L in the i-th place.

An element $\xi \in (K_2 \otimes X \cap X \otimes K_2)_{(3,0)}$ is equal to both sides of an equality

$$\operatorname{tr}\Big[\alpha A_1(t,t,t) + (t,t,t)a_1\beta\Big] = \operatorname{tr}\Big[\alpha' A_2(t,t,t) + (t,t,t)a_2\beta'\Big]$$

in TX, where

$$\begin{aligned} \alpha &: W \otimes V \to V \otimes V \otimes V \qquad & \alpha' &: V \otimes W \to V \otimes V \otimes V \\ \beta &: V \otimes V \otimes V \to W \otimes V \qquad & \beta' &: V \otimes V \otimes V \to V \otimes W \end{aligned}$$

(Here, tr means trace of a linear endomorphism with coefficients in $\mathcal{A}_{\mathcal{L}}$; e.g., in coordinates, $\operatorname{tr}\big[\alpha A_1(t,t,t)\big] = \alpha^{\ell m n}_{\alpha k} A^{\alpha}_{ij}\, t^i_\ell t^j_m t^k_n$.) It follows that

$$\alpha A_1 + a_1\beta = \alpha' A_2 + a_2\beta' \tag{6.4}$$

Now $2A_1(6.4)a_1 - A_1(6.4)a_2G - FA_2(6.4)a_1$ reads

$$\begin{aligned}2(A_1\alpha + \beta a_1) - FA_2\alpha - \beta a_2 G - A_1\alpha FG - FG\beta a_1 \\ = A_1\alpha' G + F\beta' a_1 - FA_2\alpha' G - F\beta' a_2 G\end{aligned}$$

Writing $-FG = FG - 2FG = FG - 2\rho - 2\rho\, dC$, this becomes

$$\begin{aligned}2(1-\rho)(A_1\alpha + \beta a_1) &= (A_1\alpha' + \beta a_2 - A_1\alpha F - F\beta' a_2)G \\ &\quad + F(A_2\alpha + \beta' a_1 - A_2\alpha' G - G\beta a_1) \\ &\quad + 2\rho(A_1\alpha dC + dC\beta a_1)\end{aligned}$$

Using an analogous expression of $2(1-\rho)(A_2\alpha' + \beta' a_2)$, we get

$$\begin{aligned}&2(1-\rho)(\lambda \otimes 1_X - 1_X \otimes \lambda)(\xi) \\ &\quad = 2(1-\rho)\operatorname{tr}\Big[(A_1\alpha + \beta a_1)(u,t)\Big] - 2(1-\rho)\operatorname{tr}\Big[(A_2\alpha' + \beta' a_2)(t,u)\Big] \\ &\quad = \operatorname{tr}\Big[(A_1\alpha' + \beta a_2 - A_1\alpha F - F\beta' a_2)\big(G(u,t) - (t,u)G\big)\Big] \\ &\qquad + \operatorname{tr}\Big[(A_2\alpha + \beta' a_1 - A_2\alpha' G - G\beta a_1)\big((u,t)F - F(t,u)\big)\Big] \\ &\qquad + 2\rho\operatorname{tr}\Big[(A_1\alpha dC + dC\beta a_1)(u,t)\Big] - 2\rho\operatorname{tr}\Big[(A_2\alpha' cD + cD\beta' a_2)(t,u)\Big]\end{aligned}$$

Since the coefficients of $G(u,t) - (t,u)G$, $(u,t)F - F(t,u)$, $C(u,t)$, $(u,t)d$, $D(t,u)$ and $(t,u)c$ are in K_2, this proves condition (b) in degree $(3,0)$ (noting that $\rho \neq 1$ follows from $q^4 + q^2 + 1 \neq 0$). Condition (c) is trivial. Next,

$$\begin{aligned}&2(1-\rho)\mu(\lambda \otimes 1_X - 1_X \otimes \lambda)(\xi) \\ &\quad = 2\rho\operatorname{tr}\Big[A_1\alpha dC + dC\beta a_1\Big] - 2\rho\operatorname{tr}\Big[A_2\alpha' cD + cD\beta' a_2\Big]\end{aligned}$$

but the R.H.S. vanishes, as can be seen from multiplying (6.4) on the left by $CA_1 = \omega DA_2$ and on the right by $a_1 d = \omega^2 a_2 c$. This proves condition (d) in degree $(3,0)$.

An element $\xi \in (K_2 \otimes X \cap X \otimes K_2)_{(2,1)}$ is equal to both sides of an equality

$$\begin{aligned}&\operatorname{tr}\Big[\alpha A_1(t,t,u) + (t,t,u)a_1\beta\Big] + \operatorname{tr}\Big[\gamma C_1(u,t,t) + (u,t,t)d_1\delta\Big] \\ &\quad + \operatorname{tr}\Big[\varepsilon(u,t,t)F_1 - \varepsilon F_1(t,u,t)\Big] \\ &= \operatorname{tr}\Big[\alpha' A_2(u,t,t) + (u,t,t)a_2\beta'\Big] + \operatorname{tr}\Big[\gamma' D_2(t,t,u) + (t,t,u)c_2\delta'\Big] \\ &\quad + \operatorname{tr}\Big[\varepsilon'(t,t,u)G_2 - \varepsilon' G_2(t,u,t)\Big]\end{aligned}$$

where

$$\begin{aligned}\alpha &: W \otimes W \to V \otimes V \otimes W & \alpha' &: W \otimes W \to W \otimes V \otimes V \\ \beta &: V \otimes V \otimes W \to W \otimes W & \beta' &: W \otimes V \otimes V \to W \otimes W \\ \gamma &: V \to W \otimes V \otimes V & \gamma' &: V \to V \otimes V \otimes W \\ \delta &: W \otimes V \otimes V \to V & \delta' &: V \otimes V \otimes W \to V \\ \varepsilon &: W \otimes V \otimes V \to V \otimes W \otimes V & \varepsilon' &: V \otimes V \otimes W \to V \otimes W \otimes V\end{aligned}$$

It follows that

$$\alpha A_1 + a_1\beta = \gamma' D_2 + c_2\delta' + G_2\varepsilon' \tag{6.5a}$$

$$\alpha' A_2 + a_2\beta' = \gamma C_1 + d_1\delta + F_1\varepsilon \tag{6.5b}$$

$$\varepsilon F_1 = \varepsilon' G_2 \tag{6.5c}$$

Noting that $G_1 a_2 = F_2 a_1$, (6.5c)$G_1 a_2$ reads

$$\varepsilon a_2 + \omega^2 \varepsilon d_1 B = \varepsilon' a_1 + \varepsilon' c_2 B$$

whereas A_1(6.5a)a_1 and A_2(6.5b)a_2 read

$$A_1\alpha + \beta a_2 = A_1\gamma' B + b\beta' a_1 + A_1 G_2 \varepsilon' a_1$$
$$A_2\alpha' + \beta' a_2 = \omega^2 A_2 \gamma B + \omega b\delta a_2 + A_2 F_1 \varepsilon a_2$$

Since $A_1 G_2 = A_2 F_1$, we therefore get

$$\begin{aligned}(\lambda\otimes 1_X - 1_X\otimes\lambda)(\xi) &= \operatorname{tr}\Big[(A_1\alpha + \beta a_1 - A_2\alpha' - \beta' a_2)(u,u)\Big]\\ &= \operatorname{tr}\Big[(A_1\gamma' - \omega^2 A_2\gamma + \omega^2 A_1 G_2 \varepsilon d_1 - A_2 F_1 \varepsilon' c_2)B(u,u)\Big]\\ &\quad + \operatorname{tr}\Big[(u,u)b(\delta' a_1 - \omega\delta a_2)\Big]\end{aligned}$$

Since the coefficients of $B(u,u)$ and $(u,u)b$ are in K_2, this proves condition (b) in degree $(2,1)$. Next,

$$\begin{aligned}&\lambda(\lambda\otimes 1_X - 1_X\otimes\lambda)(\xi)\\ &\quad= \operatorname{tr}\Big[(BA_1\gamma' - \omega^2 BA_2\gamma + \omega^2 BA_1 G_2\varepsilon d_1 - BA_2 F_1 \varepsilon' c_2 + \delta' a_1 b - \delta a_2 b)\, t\Big]\\ &\quad= \operatorname{tr}\Big[\big(BA_1\gamma' - \omega^2 BA_2\gamma + \delta' a_1 b - \omega\delta a_2 b + \rho(D_1 + C_2)(\omega\varepsilon d_1 - \omega^2\varepsilon' c_2)\big)\, t\Big]\end{aligned}$$

On the other hand,

$$-(\mu\otimes 1_X - 1_X\otimes\mu)(\xi) = \operatorname{tr}\Big[(D_2\gamma' + \delta' c_2 - C_1\gamma - \delta d_1)\, t\Big]$$

Now (6.5c)c_1 and (6.5c)d_2 read

$$\varepsilon d_1 = \omega^2 \varepsilon' G_2 c_1 = \omega\varepsilon' a_1 b$$
$$\varepsilon' c_2 = \omega\varepsilon F_1 d_2 = \varepsilon a_2 b$$

so $(BA_1 - D_2)$(6.5a)$(a_1 b - c_2)$ reads

$$0 = (1-\kappa)(BA_1\gamma' - D_2\gamma' + \delta' a_1 b - \delta' c_2) + \big(\rho D_1 + (\rho - 1)C_2\big)(\omega\varepsilon d_1 - \omega^2\varepsilon' c_2)$$

and $(\omega^2 BA_2 - C_1)$(6.5b)$(\omega a_2 b - d_1)$ reads

$$0 = (1-\kappa)(\omega^2 BA_2\gamma - C_1\gamma + \omega\delta a_2 b - \delta d_1) + \big(\rho C_2 + (\rho-1)D_1\big)(\omega^2\varepsilon' c_2 - \omega\varepsilon d_1)$$

Subtracting these two equalities and dividing by $1-\kappa = (2\rho - 1)/\rho$, we get

$$\begin{aligned}0 &= BA_1\gamma' - D_2\gamma' - \omega^2 BA_2\gamma + C_1\gamma + \delta' a_1 b - \delta' c_2 - \omega\delta a_2 b + \delta d_1\\ &\quad + \rho(D_1 + C_2)(\omega\varepsilon d_1 - \omega^2\varepsilon' c_2)\end{aligned}$$

(Note that $\kappa \neq 1$ follows from $q^4 \neq -1$.) This proves condition (c) in degree $(2,1)$. Condition (d) is trivial. □

Corollary 6.3. *If $\mathcal{L}$ is not elliptic,* $\dim \mathcal{A}^{(n)} = \sum_{k+\ell\leq n} (d_{(k,\ell)})^2$.

Proof. Use proposition 6.2 and corollary 5.4 (applied to $\mathcal{M}_{\mathcal{L}}$ and to $\mathcal{N}_{\mathcal{L}}$). □

7. A key endomorphism

Let

$$R := q - (q + q^{-1})\, aA \qquad R^* := q^{-1} - (q^{-1} + q)\, bB$$

(We have already used R in the proof of proposition 3.2.) Fix $(k, \ell) \in \mathbf{N}^2$ and define the following endomorphisms of $V^{\otimes k} \otimes W^{\otimes \ell}$:

$$\begin{aligned}
R_i &:= 1_{V^{\otimes(k-i-1)}} \otimes R \otimes 1_{V^{\otimes(i-1)}} \otimes 1_{W^{\otimes \ell}} && 1 \le i \le k-1 \\
X &:= 1_{V^{\otimes(k-1)}} \otimes cD \otimes 1_{W^{\otimes(\ell-1)}} \\
R_i^* &:= 1_{V^{\otimes k}} \otimes 1_{W^{\otimes(i-1)}} \otimes R^* \otimes 1_{W^{\otimes(\ell-i-1)}} && 1 \le i \le \ell-1
\end{aligned}$$

Proposition 7.1. *The ring of endomorphisms of $V^{\otimes k} \otimes W^{\otimes \ell}$ generated by the R_i's, the R_i^*'s and X contains an element P such that the kernel of the multiplication $V^{\otimes k} \otimes W^{\otimes \ell} \to V_{(k,\ell)}$ is contained in* $\operatorname{Ker} P$ *and contains* $\operatorname{Im}(P - 1)$.

Proof. First, the following relations are easily obtained from (3.2) and (3.11):

$$\begin{aligned}
R_i R_j &= R_j R_i \quad & R_i^* R_j^* &= R_j^* R_i^* \qquad && \text{if } |i - j| \ge 2 \\
X R_i &= R_i X & X R_i^* &= R_i^* X && \text{if } i \ge 2
\end{aligned}$$

$$\begin{aligned}
R_i R_j^* &= R_j^* R_i & X^2 &= \kappa X \\
R_i R_{i+1} R_i &= R_{i+1} R_i R_{i+1} & X R_1 X &= q^3 X \\
R_i^* R_{i+1}^* R_i^* &= R_{i+1}^* R_i^* R_{i+1}^* & X R_1^* X &= q^{-3} X \\
(R_i - q)(R_i + q^{-1}) &= 0 & X R_1 R_1^* X R_1 &= X R_1 R_1^* X R_1^{*-1} \\
(R_i^* - q^{-1})(R_i^* + q) &= 0 & R_1^* X R_1 R_1^* X &= R_1^{-1} X R_1 R_1^* X
\end{aligned} \tag{7.1}$$

Note that these are the relations appearing in [15, def. 2.1]. (*Warning:* the index convention used here differs from [15], and also from that in the proof of proposition 6.2.) In the sequel, we shall use the letters a, b, c, d as indices; this should not cause confusion with the maps (3.1).

Let $T_b^a := R_a^* \dots R_1^* X R_1 \dots R_b$ (for $0 \le a \le \ell - 1$, $0 \le b \le k - 1$). Using (7.1), we get

$$\begin{aligned}
T_b^a R_p &= \begin{cases} R_{p+1} T_b^a & \text{if } p \le b-1 \\ (q - q^{-1}) T_b^a + T_{b-1}^a & \text{if } p = b \\ T_{b+1}^a & \text{if } p = b+1 \\ R_p T_b^a & \text{if } p \ge b+2 \end{cases} \\
R_p^* T_b^a &= \begin{cases} T_b^a R_{p+1}^* & \text{if } p \le a-1 \\ (q^{-1} - q) T_b^a + T_b^{a-1} & \text{if } p = a \\ T_b^{a+1} & \text{if } p = a+1 \\ T_b^a R_p^* & \text{if } p \ge a+2 \end{cases}
\end{aligned} \tag{7.2}$$

and

$$
\begin{aligned}
&T_b^a T_d^c = R_1^{-1} T_b^{c-1} T_d^a && \text{if } a \geq c \geq 1 \text{ and } b \geq 1\\
&T_b^a T_d^c = T_d^a T_{b-1}^c R_1^{*-1} && \text{if } d \geq b \geq 1 \text{ and } c \geq 1\\
&T_b^a T_d^0 = q^3\, R_2 \dots R_b\, T_d^a && \text{if } b \geq 1\\
&T_0^a T_d^c = q^{-3}\, T_d^a\, R_c^* \dots R_2^* && \text{if } c \geq 1\\
&T_0^a T_d^0 = \kappa\, T_d^a
\end{aligned} \tag{7.3}
$$

Define $S = S_k$ as in (3.7), and define $S^* = S_\ell^*$ similarly (replacing R_i by R_i^* and q by q^{-1}). For $m \geq 1$, let

$$U_m := \sum_{\substack{0 \leq a_1 < \dots < a_m \leq \ell-1 \\ k-1 \geq b_1 > \dots > b_m \geq 0}} q^{(b_1-a_1)+\dots+(b_m-a_m)}\, T_{b_1}^{a_1} \dots T_{b_m}^{a_m}$$

(Note that $U_m = 0$ if $m > \min(k,\ell)$.) Consider a linear combination

$$P := \alpha_0\, SS^* + \sum_{m=1}^{\min(k,\ell)} \alpha_m\, SU_m S^*$$

Since $S^* R_i^* = q^{-1} S^*$, we have $PR_i^* = q^{-1} P$ $(1 \leq i \leq \ell-1)$. It also follows from (7.2) and (3.8) that $SU_m S^*\, R_i = q\, SU_m S^*$ for $1 \leq i \leq k-1$, so $PR_i = q\, P$.

Let $\tilde{S}^*$ be $S_{\ell-1}^*$ acting on the $\ell-1$ rightmost factors of $V^{\otimes k} \otimes W^{\otimes \ell}$; we still have $\tilde{S}^* R_i^* = q^{-1} \tilde{S}^*$ for $i \geq 2$. Let also U_m' be the sum of all terms of U_m in which $b_m = 0$, and let $U_m'' = U_m - U_m'$. By an equality analogous to (3.9), we have

$$S^* X = (T_0^0 + q^{-1}\, T_0^1 + \dots + q^{-(\ell-1)}\, T_0^{\ell-1}) \tilde{S}^* \tag{7.4}$$

From (7.2), (7.3) and (3.8), it follows that

$$
\begin{aligned}
SU_m'(T_0^0 + q^{-1}\, T_0^1 + \dots + q^{-(\ell-1)}\, T_0^{\ell-1}) \tilde{S}^* &= q^2\, [\ell+2]_{q^{-2}}\, SU_m' \tilde{S}^*\\
SU_m''(T_0^0 + q^{-1}\, T_0^1 + \dots + q^{-(m-1)}\, T_0^{m-1}) \tilde{S}^* &= q^4\, [k-m]_{q^2}\, SU_m' \tilde{S}^*\\
SU_m''(q^{-m}\, T_0^m + q^{-(m+1)}\, T_0^{m+1} + \dots + q^{-(\ell-1)}\, T_0^{\ell-1}) \tilde{S}^* &= [m+1]_{q^{-2}}\, SU_{m+1}' \tilde{S}^*
\end{aligned}
$$

Adding these three relations and using (7.4), we get

$$SU_m S^*\, X = q^{-2\ell}\, [k+\ell-m+2]_{q^2}\, SU_m' \tilde{S}^* + [m+1]_{q^{-2}}\, SU_{m+1}' \tilde{S}^*$$

It follows that for a suitable choice of the constants α_m, we have $PX = 0$. As a reminder, we have so far obtained the relations

$$PR_i = q\, P \qquad PR_i^* = q^{-1}\, P \qquad PX = 0 \tag{7.5}$$

Next, if we define the maps a_i and b_i the same way as R_i and R_i^*, respectively, then P is of the form

$$P = \alpha_0\, [k]_{q^2}\,!\, [\ell]_{q^{-2}}\,! + a_{k-1} * + \dots + a_1 * + c * + b_1 * + \dots + b_{\ell-1} * \tag{7.6}$$

Since $[n]_{q^2} \neq 0$ for every n, we may rescale the α_m so that $\alpha_0\, [k]_{q^2}\,!\, [\ell]_{q^{-2}}\,! = 1$. Now combine (7.5), (7.6) and lemma 5.1. $\square$

8. Simple $\mathcal{A}_\mathcal{L}$-comodules

Proposition 8.1. *If $\mathcal{L}$ is a nonelliptic BQD, then $\mathcal{A}_\mathcal{L}$ is a quantum* SL(3).

Proof. In the sequel, reasonings involving a degree $(k,\ell) \in \mathbf{N}^2$ will work even in the limit cases $k=0$ and $\ell=0$, provided one drops anything involving a negative index.

The canonical $\big((V^*\otimes V)\oplus(W^*\otimes W)\big)$-comodule structure on $V\oplus W$ turns the shape algebra $\mathcal{M}_\mathcal{L}$ into an $\mathbf{N}^2$-graded (infinite-dimensional) $\mathcal{A}_\mathcal{L}$-comodule algebra, hence every $V_{(k,\ell)}$ ($(k,\ell)\in\mathbf{N}^2$) into an $\mathcal{A}_\mathcal{L}$-comodule. Note that by corollary 5.4, the dimension requirements of definition 1.1 are satisfied.

We first prove the following statements by induction over n:

A_n. All $V_{(k,\ell)}$, $k+\ell\le n$, are simple and pairwise nonisomorphic.
B_n. For every (k,ℓ), $k+\ell\le n$, we have decompositions

$$V_{(k,\ell)}\otimes V \simeq V_{(k+1,\ell)}\oplus V_{(k,\ell-1)}\oplus V_{(k-1,\ell+1)} \tag{8.1a}$$

$$V_{(k,\ell)}\otimes W \simeq V_{(k,\ell+1)}\oplus V_{(k-1,\ell)}\oplus V_{(k+1,\ell-1)} \tag{8.1b}$$

Statements A_0 and B_0 are clear.

Let $n>0$. Applying statement B_{n-1} repeatedly, we get

$$\bigoplus_{p=0}^{n}(V\oplus W)^{\otimes p}\simeq V_{(k_1,\ell_1)}\oplus\cdots\oplus V_{(k_s,\ell_s)}$$

where $k_i+\ell_i\le n$ for every i (and actually, every $V_{(k,\ell)}$, $k+\ell\le n$, appears at least once), so the coefficient space of the R.H.S. has dimension at most $\sum_{k+\ell\le n}(d_{(k,\ell)})^2$, with equality if and only if statement A_n holds. But in view of the L.H.S., this coefficient space is precisely $\mathcal{A}^{(n)}$, so by corollary 6.3, we do have the desired equality. This shows that B_{n-1} implies A_n.

Let still $n>0$ and let $k+\ell=n$. Denote by μ the multiplication in $\mathcal{M}_\mathcal{L}$ and define

$$\alpha: V_{(k-1,\ell)}\xrightarrow{(1,c)} V_{(k-1,\ell)}\otimes V\otimes W\xrightarrow{(\mu,1)} V_{(k,\ell)}\otimes W$$
$$\beta: V_{(k+1,\ell-1)}\xrightarrow{\gamma} V_{(k,\ell-1)}\otimes V\xrightarrow{(1,b)} V_{(k,\ell-1)}\otimes W\otimes W\xrightarrow{(\mu,1)} V_{(k,\ell)}\otimes W$$

where γ is the injection (provided by statement B_{n-1}) such that $\mu\gamma=1$. Consider the sequence

$$0\to V_{(k-1,\ell)}\oplus V_{(k+1,\ell-1)}\xrightarrow{\alpha\oplus\beta} V_{(k,\ell)}\otimes W\xrightarrow{\mu} V_{(k,\ell+1)}\to 0 \tag{8.2}$$

By lemma 5.1, μ is surjective. Moreover,

$$(1,C)(\alpha,1_V)=(1,C)(\mu,1,1)(1,c,1)=\mu(1,1,C)(1,c,1)=\mu$$

and

$$\begin{aligned}\mu(1,B)(\beta,1_W)&=\mu(1,B)(\mu,1,1)(1,b,1)(\gamma,1)=\mu(\mu,1)(1,1,B)(1,b,1)(\gamma,1)\\&=\omega\,\mu(\mu,1)(1,F)(\gamma,1)=\omega\,\mu(1,\mu)(1,F)(\gamma,1)\\&=\omega(q+q^{-1})^{-1}\,\mu(1,\mu)(\gamma,1)=\omega(q+q^{-1})^{-1}\,\mu(\mu,1)(\gamma,1)\\&=\omega(q+q^{-1})^{-1}\,\mu\end{aligned}$$

hence $\alpha \neq 0$ and $\beta \neq 0$. Statement A_n then implies that $\alpha \oplus \beta$ is injective. Moreover,

$$\mu\alpha = \mu(\mu, 1)(1, c) = \mu(1, \mu)(1, c) = 0$$
$$\mu\beta = \mu(\mu, 1)(1, b)\gamma = \mu(1, \mu)(1, b)\gamma = 0$$

so in view of dimensions, the sequence (8.2) is exact. Using the $\mathcal{A}$-endomorphism P provided by proposition 7.1, we may split this exact sequence; this implies (8.1b). Since (8.1a) is similar, this shows that B_{n-1} and A_n imply B_n and completes the induction.

Definition 1.1(a) is now clear. Moreover, the sum of the coefficient spaces of all the $V_{(k,\ell)}$ is equal to $\bigcup \mathcal{A}^{(n)} = \mathcal{A}_{\mathcal{L}}$, which implies definition 1.1(b).

Finally, (8.1a) implies

$$\begin{aligned} & V_{(k,\ell)} \otimes V_{(p,q)} \otimes V \\ & \quad \simeq (V_{(k,\ell)} \otimes V_{(p+1,q)}) \oplus (V_{(k,\ell)} \otimes V_{(p,q-1)}) \oplus (V_{(k,\ell)} \otimes V_{(p-1,q+1)}) \end{aligned} \tag{8.3}$$

By induction over $p + q$, $V_{(k,\ell)} \otimes V_{(p,q-1)}$, $V_{(k,\ell)} \otimes V_{(p-1,q+1)}$ and $V_{(k,\ell)} \otimes V_{(p,q)}$ decompose according to definition 1.1(c), and by (8.1), $V_{(k,\ell)} \otimes V_{(p,q)} \otimes V$ still decomposes according to it. Therefore, (8.3) implies that $V_{(k,\ell)} \otimes V_{(p+1,q)}$ does so as well. The case $V_{(k,\ell)} \otimes V_{(p,q+1)}$ is similar. □

9. Equivalence between quantum SL(3)'s and BQD's

Let $\mathcal{L} = (V, W, A, a, B, b, C, c, D, d)$ be a BQD and consider the following three transformations of $\mathcal{L}$:

- base change, i.e., conjugating A, a, B, b, C, c, D, d by some invertible linear maps $V \to V'$, $W \to W'$ (where V', W' are any vector spaces of dimension 3);
- multiplying A, a, B, b, C, c, D, d by scalars (but such that (3.2) are preserved);
- interchanging $V \leftrightarrow W$, $A \leftrightarrow B$, $a \leftrightarrow b$, $C \leftrightarrow D$, $c \leftrightarrow d$ (this still gives a BQD by (3.11)).

We call the third transformation *Dynkin flip*, and it should indeed be thought of as applying the automorphism of the Dynkin diagram of SL(3). Two BQD's are called *equivalent* if one is obtained from the other by any combination of these three transformations.

Theorem 9.1. *The correspondences $\mathcal{A} \mapsto \mathcal{L}_{\mathcal{A}}$ and $\mathcal{L} \mapsto \mathcal{A}_{\mathcal{L}}$ are inverse of each other between nonelliptic quantum* SL(3)*'s (up to Hopf algebra isomorphism) and nonelliptic BQD's (up to equivalence).*

Proof. These two correspondences are well-defined by propositions 3.1, 3.2 and 8.1. Also, if $\mathcal{L}$ is not elliptic, then $\mathcal{L}_{\mathcal{A}_{\mathcal{L}}}$ is clearly equivalent to $\mathcal{L}$.

Conversely, if $\mathcal{A}$ is a nonelliptic quantum SL(3), then the algebras $\mathcal{A}_{\mathcal{L}_{\mathcal{A}}}$ and $\mathcal{A}$ have a common generating space $\mathrm{Coeff}(t) + \mathrm{Coeff}(u)$, so both have an $\mathbf{N}$-filtration $\mathcal{A}_{\mathcal{L}_{\mathcal{A}}} = \bigcup \mathcal{A}^{(n)}$, $\mathcal{A} = \bigcup \mathcal{B}^{(n)}$. Furthermore, the defining relations (3.12) of $\mathcal{A}_{\mathcal{L}_{\mathcal{A}}}$ are also valid in $\mathcal{A}$, so there is a canonical surjection $\mathcal{A}_{\mathcal{L}_{\mathcal{A}}} \to \mathcal{A}$, which restricts to the identity on $\mathrm{Coeff}(t) + \mathrm{Coeff}(u)$, and therefore takes $\mathcal{A}^{(n)}$ onto $\mathcal{B}^{(n)}$. But

$$\dim \mathcal{A}^{(n)} = \sum_{k+\ell \leq n} (d_{(k,\ell)})^2 = \dim \mathcal{B}^{(n)}$$

where the left equality follows from corollary 6.3 and the right one from the Peter-Weyl decomposition. Thus, $\mathcal{A}_{\mathcal{L}_{\mathcal{A}}}$ is isomorphic to $\mathcal{A}$. □

10. Classification of BQD's

10.0. Strategy for the classification. Let

$$Q := c^\flat D^\sharp : V \to V$$

By (3.2a), Q is invertible with $Q^{-1} = d^\flat C^\sharp$, so by (3.2f),

$$\operatorname{tr} Q = \kappa = \operatorname{tr} Q^{-1} \tag{10.1}$$

(Note that by (4.2), Q "encodes" the square of the antipode, in the sense that $S^2(t) = QtQ^{-1}$.) Now define

$$E := C(A, 1_V) = \omega\, D(1_V, A) \qquad e := (1_V, a)c = \omega\,(a, 1_V)d \tag{10.2}$$

Let x_1, x_2, x_3 be a basis of V and y_1, y_2, y_3 a basis of W, and denote by x^1, x^2, x^3 and y^1, y^2, y^3 their respective dual bases. Write $Q(x_i) = Q_i^j x_j$, $e = e^{ijk} x_{ijk}$ and $E = E_{ijk} x^{ijk}$, where

$$x_{ijk} := x_i \otimes x_j \otimes x_k \qquad x^{ijk} := x^i \otimes x^j \otimes x^k$$

By (3.2), we have

$$e^{\ell ij} = \omega^2\, e^{ijk} Q_k^\ell \qquad E_{\ell ij} = \omega\, E_{ijk} Q_\ell^k \tag{10.3}$$

$$e^{ik\ell} E_{k\ell j} = \delta_j^i \tag{10.4}$$

Proposition 10.1. *Q is of one of the following four types:*

Type I: *Q is the identity.*

Type II: *Q has eigenvalues $q^{-2}, 1, q^2$ for some q, with $q^{2n} \neq 1$ for all n (so Q is diagonalizable).*

Type III: *Q has triple eigenvalue 1 and a 2×2 Jordan block.*

Type IV: *Q has triple eigenvalue 1 and a 3×3 Jordan block.*

Proof. Let α, β, γ be the three eigenvalues of Q.

Case 1: $\alpha = \beta = \gamma$. Then (up to base change) $Q = \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \alpha & 0 & 2 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \alpha & 2 & 2 \\ 0 & \alpha & 2 \\ 0 & 0 & \alpha \end{smallmatrix}\right)$. (The unusual normalizations of the nondiagonal cases will be convenient later.)

By (10.1), $\alpha^2 = 1$. Taking $\alpha = -1$ in (10.3) prevents e, E from satisfying (10.4) (for either value of ω and either form of Q), so we must have $\alpha = 1$. This gives types I, III and IV.

Case 2: $\alpha = \gamma \neq \beta$. Then $Q = \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \alpha & 0 & 1 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha \end{smallmatrix}\right)$.

If $\alpha^2\beta = 1$, then (10.1) implies $\alpha = \pm 1$, hence $\beta = 1$, and $\alpha = -1$. But then $\kappa = -1$, contradicting $q^2 \neq -1$. Therefore $\alpha^2\beta \neq 1$. Taking $\alpha^3 \neq 1$ in (10.3) then prevents e, E from satisfying (10.4), so we must have $\alpha^3 = 1$.

If $\alpha\beta^2 = 1$, then (10.1) implies $\alpha = 1$, $\beta = -1$, hence $\kappa = 1$, contradicting $q^4 \neq -1$. Therefore $\alpha\beta^2 \neq 1$.

Now (10.4) implies $e^{222}E_{222} = 1$, so (10.3) implies $\beta = \omega$ and $\beta = \omega^2$, hence $\beta = 1$. Then (10.1) implies $\alpha^2 = 1$, contradicting $\alpha^2\beta \neq 1$. Consequently, case 2 is impossible.

Case 3: $\alpha \neq \beta \neq \gamma \neq \alpha$. Then $Q = \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{smallmatrix}\right)$.

Subcase 3a: $\alpha\beta\gamma = 1$. Then (10.1) implies $(\alpha - 1)(\beta - 1)(\gamma - 1) = 0$, hence, say, $\beta = 1$. Then $\alpha = \gamma^{-1}$ and $\kappa = \gamma^{-1} + 1 + \gamma$. Choosing q such that $q^2 = \gamma$ (so that $\kappa = q^{-2} + 1 + q^2$) gives type II.

Subcase 3b: $\alpha\beta\gamma \neq 1$. If, say, $\beta = 1$, then (10.1) implies $\gamma = \alpha^{-1}$ or $\gamma = -\alpha$. Both are impossible ($\gamma = \alpha^{-1}$ contradicts $\alpha\beta\gamma \neq 1$ and $\gamma = -\alpha$ implies $\kappa = 1$, contradicting $q^4 \neq -1$), so we must have $\alpha \neq 1$, $\beta \neq 1$ and $\gamma \neq 1$.

If at least five of the six scalars $\alpha^2\beta$, $\alpha^2\gamma$, $\beta^2\alpha$, $\beta^2\gamma$, $\gamma^2\alpha$, $\gamma^2\beta$ are different from 1, then (10.3) prevents e, E from satisfying (10.4). Therefore, say, $\alpha^2\beta = 1$ (so that $\alpha^2\gamma \neq 1$ and $\beta^2\alpha \neq 1$) and at least one of $\beta^2\gamma$, $\gamma^2\alpha$, $\gamma^2\beta$ equals 1.

But $\beta^2\gamma = 1$ implies $\gamma = \beta^{-2} = \alpha^4$, so by (10.1), $\kappa = 0$, contradicting $q^4+q^2+1 \neq 0$. Similarly, $\gamma^2\alpha = 1$ is impossible.

If $\gamma^2\beta = 1$, then $\gamma = -\alpha$, and (10.1) implies $\alpha^4 = 1$, hence $\beta = \pm 1$, so $\kappa = \pm 1$, both of which we already know to be excluded. Consequently, subcase 3b is impossible. □

If Q is of type X (X=I, II, III or IV), we call the BQD of type X if $\omega = 1$, and of type X' if $\omega^2 + \omega + 1 = 0$.

The strategy to classify BQD's of type I, I', II, II', III, III', IV and IV' will be as follows:

- In types II and II', take the given q; in all other types, set $q = 1$.
- The choice of a "nondegenerate" $c : \mathbf{C} \to V \otimes W$ being arbitrary (changing it amounts to a base change in W), deduce D from $Q = c^\flat D^\sharp$, then C, d from (3.2a).
- Choose $e : \mathbf{C} \to V \otimes V \otimes V$ satisfying (10.3), working modulo the stabilizer $\mathrm{Stab}(Q)$ of Q in $\mathrm{GL}(V)$.
- Determine all possible $E : V \otimes V \otimes V \to \mathbf{C}$ satisfying (10.3), (10.4) and (3.2gh) (where A, a are deduced from (10.2)).
- Reduce the possible forms for E modulo $\mathrm{Stab}(Q, e)$.

In some cases, it may be useful to swap the last two steps.

We shall allow ourselves to satisfy (10.4) only up to a nonzero scalar, adapting (3.2b) and (3.2gh) accordingly.

Finally, we leave out the details of matrix computations, which the reader can easily recheck using any standard computer algebra package. (The author used Maple V Release 3.)

10.1. **Type I.** We take

$$\begin{aligned} c &= x_1 \otimes y_1 + x_2 \otimes y_2 + x_3 \otimes y_3 & \qquad d &= y_1 \otimes x_1 + y_2 \otimes x_2 + y_3 \otimes x_3 \\ C &= y^1 \otimes x^1 + y^2 \otimes x^2 + y^3 \otimes x^3 & \qquad D &= x^1 \otimes y^1 + x^2 \otimes y^2 + x^3 \otimes y^3 \end{aligned}$$

We have $\mathrm{Stab}(Q) = \mathrm{GL}(V)$. By (10.3), $e = \lambda + s$ and $E = \Lambda + S$, where λ, Λ are totally antisymmetric and s, S totally symmetric. We view s (resp. S) as a polynomial function on V^* (resp. V), viz.

$$\begin{aligned} s &= e^{111}x_1^3 + e^{222}x_2^3 + e^{333}x_3^3 + 3(e^{123} + e^{132})x_1x_2x_3 \\ &\quad + 3e^{112}x_1^2x_2 + 3e^{113}x_1^2x_3 + 3e^{221}x_2^2x_1 + 3e^{223}x_2^2x_3 \\ &\quad + 3e^{331}x_3^2x_1 + 3e^{332}x_3^2x_2 \\ S &= E_{111}X_1^3 + E_{222}X_2^3 + E_{333}X_3^3 + 3(E_{123} + E_{132})X_1X_2X_3 \\ &\quad + 3E_{112}X_1^2X_2 + 3E_{113}X_1^2X_3 + 3E_{221}X_2^2X_1 + 3E_{223}X_2^2X_3 \\ &\quad + 3E_{331}X_3^2X_1 + 3E_{332}X_3^2X_2 \end{aligned}$$

where we write X_i instead of x^i, to improve legibility. We shall also consider s as a cubic curve in the projective plane $\mathbb{P}V^*$ and S as one in $\mathbb{P}V$.

We now examine the different normal forms for s modulo $\mathrm{GL}(V)$ if $\lambda = 0$, and modulo $\mathrm{Stab}(\lambda) = \mathrm{SL}(V)$ if $\lambda \neq 0$ (see, e.g., [16, §I.7]).

The following cases are possible for $\lambda \neq 0$, $\Lambda \neq 0$ (in cases I.a–I.g, we normalize so that $\lambda_{123} = 1$, $\Lambda^{321} = 1$).

Case I.a: $s = 0$, $S = 0$. The resulting Hopf algebra is that of functions on the (ordinary) group SL(3).
Case I.b: $s = x_1^3$ and $S = 0$.
Case I.c: $s = x_1^3$ and $S = X_3^3$: s is a triple line ℓ^3 in $\mathbb{P}V^*$ and S is a triple line p^3 in $\mathbb{P}V$, with p (viewed as a point in $\mathbb{P}V^*$) lying on ℓ.
Case I.d: $s = x_1^2 x_2$ and $S = X_2 X_3^2$: $s = \ell^2 \cup \ell'$ and $S = p^2 \cup p'$, with $p = \ell \cap \ell'$ and $p' \in \ell$.
Case I.e: $s = \alpha\, x_1 x_2 x_3$ and $S = \alpha\, X_1 X_2 X_3$, with $\alpha \neq 0, 1, -1$: $s = \ell \cup \ell' \cup \ell''$ and $S = p \cup p' \cup p''$, with $\ell \cap \ell' \cap \ell'' = \emptyset$, $p = \ell' \cap \ell'$, $p' = \ell'' \cap \ell$, $p'' = \ell \cap \ell'$.
Case I.f: $s = 6i\sqrt{3}\, x_1(x_1^2 + x_2 x_3)$ and $S = 6i\sqrt{3}\, X_1 X_2 X_3$, where $i^2 = -1$: same configuration as case I.e, but $s = \mathcal{C} \cup \ell$ and $S = p \cup p' \cup p''$, with $\mathcal{C}$ a (nondegenerate) conic tangent to ℓ' at p'' and to ℓ'' at p'.
Case I.g: $s = 6i\sqrt{3}\, x_1(x_1^2 + x_2 x_3)$ and $S = 6i\sqrt{3}\, X_3(X_3^2 + X_1 X_2)$: same configuration as case I.f, but $s = \mathcal{C} \cup \ell$ and $S = \mathcal{C}' \cup p'$, with $\mathcal{C}'$ a conic tangent to p at ℓ'' and to p'' at ℓ.
Case I.h: $e = \alpha(x_{123} + x_{231} + x_{312}) + \beta(x_{132} + x_{213} + x_{321}) + \gamma(x_{111} + x_{222} + x_{333})$ and $E = \alpha'(x^{123} + x^{231} + x^{312}) + \beta'(x^{132} + x^{213} + x^{321}) + \gamma'(x^{111} + x^{222} + x^{333})$, with $\gamma \neq 0$, $\gamma' \neq 0$, $\gamma^3 + (\alpha + \beta)^3 \neq 0$, ${\gamma'}^3 + (\alpha' + \beta')^3 \neq 0$ (so s and S are elliptic curves), $\alpha \neq \beta$, $\alpha' \neq \beta'$, $\alpha\alpha' + \beta\beta' + \gamma\gamma' \neq 0$ and

$$\begin{cases} \alpha^2{\alpha'}^2 + \beta^2{\beta'}^2 + \gamma^2{\gamma'}^2 - 2\alpha\alpha'\beta\beta' - 2\alpha\alpha'\gamma\gamma' - 2\beta\beta'\gamma\gamma' = 0 \\ \alpha^2\beta'\gamma' + \beta^2\alpha'\gamma' + \gamma^2\alpha'\beta' = 0 \\ {\alpha'}^2\beta\gamma + {\beta'}^2\alpha\gamma + {\gamma'}^2\alpha\beta = 0 \end{cases} \tag{10.5}$$

(In this case, we refrain from normalizing the antisymmetric parts $\alpha - \beta$ and $\alpha' - \beta'$, to keep (10.5) homogeneous.) Note that there are solutions, e.g., $\alpha = \alpha' = 0$ and $\beta = \beta' = \gamma = \gamma' \neq 0$.

Question. Can conditions (10.5) be described geometrically in terms of the elliptic curves s and S?

(The cases where s is a cusp curve, a node curve, a conic with a tangent line or three intersecting lines cannot occur.)

The case $\lambda \neq 0$, $\Lambda = 0$ (or vice-versa) is impossible.

There is only one possible case with $\lambda = 0$, $\Lambda = 0$.

Case I.e*: $\lambda = 0$, $\Lambda = 0$, $s = x_1 x_2 x_3$ and $S = X_1 X_2 X_3$. (The geometry is similar to case I.e.)

10.2. **Type I'.** We take C, D, c, d as for type I. Define

$$z_{ijk} := x_{ijk} + \omega x_{jki} + \omega^2 x_{kij} \qquad z^{ijk} := x^{ijk} + \omega x^{kij} + \omega^2 x^{jki}$$

Let Γ_ω be the subspace of $V^{\otimes 3}$ spanned by all z_{ijk} and Γ_ω^* that of $V^{*\otimes 3}$ spanned by all z^{ijk}. Now (10.3) means that $e \in \Gamma_\omega$ and $E \in \Gamma_\omega^*$.

Note that Γ_ω is a sub-GL(V)-module of $V^{\otimes 3}$, isomorphic to the GL(V)-module $\mathfrak{sl}(V) = \{X \in \mathrm{Lin}(V,V) \mid \operatorname{tr} X = 0\}$. We use the isomorphism given by

$$\begin{pmatrix} t_1 & u_1 & u_3 \\ v_1 & t_2 - t_1 & u_2 \\ v_3 & v_1 & -t_2 \end{pmatrix} \mapsto \begin{array}{l} t_1(z_{123} + z_{213}) + t_2(z_{231} + z_{321}) \\ \quad - u_1 z_{113} - u_2 z_{221} + u_3 z_{112} \\ \quad + v_1 z_{223} + v_2 z_{331} - v_3 z_{332} \end{array}$$

to view e as an element of $\mathfrak{sl}(V)$. We do the same for E, using the isomorphism

$$\begin{pmatrix} t_1' & u_1' & u_3' \\ v_1' & t_2' - t_1' & u_2' \\ v_3' & v_1' & -t_2' \end{pmatrix} \mapsto \begin{array}{l} t_1'(z^{123} + z^{213}) + t_2'(z^{231} + z^{321}) \\ \quad - u_1' z^{223} - u_2' z^{331} + u_3' z^{332} \\ \quad + v_1' z^{113} + v_2' z^{221} - v_3' z^{112} \end{array}$$

between Γ_ω^* and $\mathfrak{sl}(V)$.

Using $\mathrm{Stab}(Q) = \mathrm{GL}(V)$, we reduce e to a Jordan normal form. Since $\operatorname{tr} e = 0$ and $e \neq 0$, there are two cases.

Case I'.a: e is diagonal, i.e. (after rescaling), $u_i = v_i = 0$ $(i = 1,2,3)$, $t_1 = 1$, $t_2 \neq 2$, $t_2 \neq \frac{1}{2}$. Now (10.4) reads

$$u_1' = u_2' = v_1' = v_2' = 0 \qquad u_3'(t_2 + 1) = v_3'(t_2 + 1) = 0$$
$$(t_2 - 2)t_1' + (1 - 2t_2)t_2' \neq 0$$

There are two subcases.

- If $t_2 \neq -1$, then $u_3' = v_3' = 0$, and (3.2gh) are equivalent to $t_2 t_1' = t_2'$. Rescaling e, we get $t_1' = 1$, $t_2' = t_2$.
- If $t_2 = -1$, then (3.2gh) are equivalent to $u_3' = v_3' = 0$, $t_2' = -t_1'$. Rescaling E, we get $t_1' = 1$, $t_2' = -1$.

Combining these subcases gives a 1-parameter family

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t-1 & 0 \\ 0 & 0 & -t \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t-1 & 0 \\ 0 & 0 & -t \end{pmatrix}$$

with the condition $t^2 - t + 1 \neq 0$.

Case I'.b: e has a 2×2 Jordan block, i.e. (after rescaling), $v_1 = v_2 = v_3 = 0$, $u_1 = u_2 = 0$, $u_3 = 1$, $t_1 = -t_2 = 1$. Now (10.4) reads

$$v_1' = v_2' = v_3' = 0 \qquad u_1' = u_2' = 0 \qquad t_1' = -t_2' \neq 0$$

and (3.2gh) are equivalent to $t_1' = -u_3'$. Rescaling E, we obtain the solution

$$e = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

10.3. **Type II.** We take

$$c = q^{-1}\, x_1 \otimes y_1 + x_2 \otimes y_2 + q\, x_3 \otimes y_3 \qquad d = q\, y_1 \otimes x_1 + y_2 \otimes x_2 + q^{-1}\, y_3 \otimes x_3$$
$$C = q\, y^1 \otimes x^1 + y^2 \otimes x^2 + q^{-1}\, y^3 \otimes x^3 \qquad D = q^{-1}\, x^1 \otimes y^1 + x^2 \otimes y^2 + q\, x^3 \otimes y^3$$

By (10.3), e and E are of the form

$$e = \alpha(x_{123} + q^2\, x_{231} + q^2\, x_{312}) - \beta(x_{132} + x_{213} + q^2\, x_{321}) + \gamma x_{222}$$
$$E = \alpha'(x^{123} + q^2\, x^{231} + q^2\, x^{312}) - \beta'(x^{132} + x^{213} + q^2\, x^{321}) + \gamma' x^{222}$$

Condition (10.4) then reads

$$(q^2 - 1)(q^2 \alpha\alpha' - \beta\beta') + \gamma\gamma' = 0 \qquad \alpha\alpha' + \beta\beta' \neq 0$$

Inspecting (3.2g) $-$ ${}^{\tau}$(3.2h) shows that we must have $\gamma\gamma' = 0$, so up to Dynkin flip (cf. section 9), we may assume $\gamma' = 0$. Now (3.2gh) become equivalent to $\gamma(q^4\alpha^3 - \beta^3) = 0$. We therefore have two cases.

Case II.a: If $\gamma = 0$, we normalize e, E so that $\alpha = \alpha' = 1$. This gives a 2-parameter family, namely the Artin-Schelter-Tate quantum $SL(3)$'s [2] (or rather, their quantum GL(3)'s having a central quantum determinant), where (in the notation of [2]) $p_{21} = p_{32} = \beta$, $p_{31} = \beta'$ and $\lambda = \frac{1}{\beta\beta'} = \frac{1}{q^2}$. The standard quantum SL(3) [10] is obtained as a particular case, when $\beta = \beta' = q$.

Case II.b: If $\gamma \neq 0$, then $\beta = p^4\alpha$, with $p^3 = q$. We first normalize e, E so that $\alpha = p^{-2}$, $\beta = p^2$, $\alpha' = p^{-1}$ and $\beta' = p$, then we use $\mathrm{Stab}(Q) = \{\left(\begin{smallmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \nu\end{smallmatrix}\right)\}$ to get $\gamma = q^2 - 1$. This gives a 1-parameter family, namely the Cremmer-Gervais quantum SL(3) as described in [12].

10.4. **Type II'.** We take C, D, c, d as for type II. By (10.3), e and E are of the form

$$\begin{aligned} e &= \alpha(x_{123} + \omega q^2\, x_{231} + \omega^2 q^2\, x_{312}) - \beta(x_{132} + \omega q^2\, x_{321} + \omega^2\, x_{213}) \\ E &= \alpha'(x^{123} + \omega q^2\, x^{312} + \omega^2 q^2\, x^{231}) - \beta'(x^{132} + \omega\, x^{213} + \omega^2 q^2\, x^{321}) \end{aligned}$$

Condition (10.4) then reads

$$\beta\beta' = q^2\alpha\alpha' \qquad \alpha\alpha' + \beta\beta' \neq 0$$

Case II'.a: (unique case of this type) Rescaling e, E, we get $\alpha = \alpha' = 1$ and $\beta' = \frac{q^2}{\beta}$. Since conditions (3.2gh) are automatically fulfilled, this gives a 2-parameter family.

10.5. **Type III.** We take

$$\begin{aligned} c &= x_1 \otimes y_1 + x_2 \otimes y_2 + x_3 \otimes y_3 + x_1 \otimes y_3 \\ C &= y^1 \otimes x^1 + y^2 \otimes x^2 + y^3 \otimes x^3 - y^1 \otimes x^3 \\ d &= y_1 \otimes x_1 + y_2 \otimes x_2 + y_3 \otimes x_3 - y_3 \otimes x_1 \\ D &= x^1 \otimes y^1 + x^2 \otimes y^2 + x^3 \otimes y^3 + x^3 \otimes y^1 \end{aligned}$$

By (10.3), e and E are of the form

$$\begin{aligned} e = {} & \alpha(x_{123} + x_{231} + x_{312} - x_{321} - x_{213} - x_{132}) - 2\alpha\, x_{121} \\ & + \beta\, x_{111} + \gamma(x_{112} + x_{121} + x_{211}) + \delta(x_{122} + x_{212} + x_{221}) + \varepsilon\, x_{222} \\ E = {} & \alpha'(x^{321} + x^{213} + x^{132} - x^{123} - x^{231} - x^{312}) - 2\alpha' x^{323} \\ & + \beta' x^{333} + \gamma'(x^{332} + x^{323} + x^{233}) + \delta'(x^{322} + x^{232} + x^{223}) + \varepsilon' x^{222} \end{aligned}$$

Now (10.4) reads

$$\delta\delta' = \delta\varepsilon' = \varepsilon\delta' = \varepsilon\varepsilon' = 0 \qquad \alpha\alpha' \neq 0$$

We normalize to $\alpha = \alpha' = 1$.

If $\varepsilon = \varepsilon' = 0$, then (up to Dynkin flip) $\delta' = 0$. Now (3.2gh) is equivalent to

$$\delta = 0 \qquad (\gamma + \gamma' - 1)(\gamma + \gamma' - 2) = 0$$

Note also that if $\gamma \neq \frac{2}{3}$ (resp. $\gamma' \neq \frac{2}{3}$), then we may use $\mathrm{Stab}(Q) = \{\left(\begin{smallmatrix} \lambda & \sigma & \nu \\ 0 & \mu & \sigma' \\ 0 & 0 & \lambda \end{smallmatrix}\right)\}$ to get $\beta = 0$ (resp. $\beta' = 0$). Working up to $\mathrm{Stab}(Q)$ (and up to Dynkin flip) now leads to four cases (two of which are 1-parameter families).

Case III.a: $(\alpha, \beta, \gamma, \delta, \varepsilon) = (1, 0, \gamma, 0, 0)$ and $(\alpha', \beta', \gamma', \delta', \varepsilon') = (1, 0, 1-\gamma, 0, 0)$.
Case III.a*: $(\alpha, \beta, \gamma, \delta, \varepsilon) = (1, 1, \frac{2}{3}, 0, 0)$ and $(\alpha', \beta', \gamma', \delta', \varepsilon') = (1, 0, \frac{1}{3}, 0, 0)$.
Case III.b: $(\alpha, \beta, \gamma, \delta, \varepsilon) = (1, 0, \gamma, 0, 0)$ and $(\alpha', \beta', \gamma', \delta', \varepsilon') = (1, 0, 2-\gamma, 0, 0)$.
Case III.b*: $(\alpha, \beta, \gamma, \delta, \varepsilon) = (1, 1, \frac{2}{3}, 0, 0)$ and $(\alpha', \beta', \gamma', \delta', \varepsilon') = (1, 0, \frac{4}{3}, 0, 0)$.

If one of $\varepsilon, \varepsilon'$ is nonzero, then up to Dynkin flip, we may assume $\varepsilon \neq 0$. If follows that $\delta' = \varepsilon' = 0$. Using $\mathrm{Stab}(Q)$, we may get $\varepsilon = 1$ and $\delta = 0$. Now (3.2gh) is equivalent to

$$\beta = 0 \qquad \gamma' = \frac{1}{2}(1+\gamma) \qquad (\gamma + \gamma' - 1)(\gamma + \gamma' - 2) = 0$$

This leads to two further cases.

Case III.c: If $\gamma = \frac{1}{3}$, we have a 1-parameter family

$$(\alpha, \beta, \gamma, \delta, \varepsilon) = (1, 0, \frac{1}{3}, 0, 1) \qquad (\alpha', \beta', \gamma', \delta', \varepsilon) = (1, \beta', \frac{2}{3}, 0, 0)$$

Case III.c*: If $\gamma = 1$, we may still use $\mathrm{Stab}(Q)$ to get $\beta' = 0$. This gives the solution

$$(\alpha, \beta, \gamma, \delta, \varepsilon) = (1, 0, 1, 0, 1) \qquad (\alpha', \beta', \gamma', \delta', \varepsilon') = (1, 0, 1, 0, 0)$$

10.6. **Type III'.** We take C, D, c, d as for type III. By (10.3), e and E are of the form

$$\begin{aligned}
e &= \alpha(x_{123} + \omega\, x_{231} + \omega^2\, x_{312} - x_{321} - \omega\, x_{213} - \omega^2\, x_{132}) - 2\alpha\, x_{121} \\
&\quad + \beta\, x_{111} + \frac{\omega - 1}{2}\beta(x_{113} + \omega\, x_{131} + \omega^2\, x_{311}) \\
&\quad + \gamma(x_{112} + \omega\, x_{121} + \omega^2\, x_{211}) + \delta(x_{122} + \omega\, x_{221} + \omega^2\, x_{212}) \\
E &= \alpha'(x^{321} + \omega^2\, x^{213} + \omega\, x^{132} - x^{123} - \omega^2\, x^{231} - \omega\, x^{312}) - 2\alpha' x^{323} \\
&\quad + \beta' x^{333} + \frac{\omega^2 - 1}{2}\beta'(x^{331} + \omega^2\, x^{313} + \omega\, x^{133}) \\
&\quad + \gamma'(x^{332} + \omega^2\, x^{323} + \omega\, x^{233}) + \delta'(x^{322} + \omega^2\, x^{223} + \omega\, x^{232})
\end{aligned}$$

Now (10.4) reads

$$\begin{aligned}
3\alpha\beta' &= 2(1-\omega)\delta\alpha' &\qquad 3\beta\beta' &= 4\delta\delta' \\
3\alpha'\beta &= 2(1-\omega^2)\delta'\alpha &\qquad \alpha\alpha' &\neq 0
\end{aligned}$$

It follows that $\delta\delta' = 0$, so up to Dynkin flip, we may assume that $\delta' = 0$. Rescaling e, E so that $\alpha = \alpha' = 1$, we get $\beta = 0$. Using $\mathrm{Stab}(Q)$, we may get $\delta = 0$, so $\beta' = 0$. Now (3.2gh) are equivalent to

$$(\gamma + \omega\gamma' - \omega^2)(\gamma + \omega\gamma' - 2\omega^2) = 0$$

We therefore have two 1-parameter families.

Case III'.a: $(\alpha, \beta, \gamma, \delta) = (1, 0, \gamma, 0)$ and $(\alpha', \beta', \gamma', \delta') = (1, 0, \omega - \omega^2\gamma, 0)$
Case III'.b: $(\alpha, \beta, \gamma, \delta) = (1, 0, \gamma, 0)$ and $(\alpha', \beta', \gamma', \delta') = (1, 0, 2\omega - \omega^2\gamma, 0)$

10.7. **Type IV.** We take

$$\begin{aligned}
c &= x_1 \otimes y_1 + x_2 \otimes y_2 + x_3 \otimes y_3 + x_1 \otimes y_2 + x_2 \otimes y_3 + \tfrac{1}{2}\, x_1 \otimes y_3 \\
C &= y^1 \otimes x^1 + y^2 \otimes x^2 + y^3 \otimes x^3 - y^1 \otimes x^2 - y^2 \otimes x^3 + \tfrac{1}{2}\, y^1 \otimes x^3 \\
d &= y_1 \otimes x_1 + y_2 \otimes x_2 + y_3 \otimes x_3 - y_2 \otimes x_1 - y_3 \otimes x_2 + \tfrac{1}{2}\, y_3 \otimes x_1 \\
D &= x^1 \otimes y^1 + x^2 \otimes y^2 + x^3 \otimes y^3 + x^2 \otimes y^1 + x^3 \otimes y^2 + \tfrac{1}{2}\, x^3 \otimes y^1
\end{aligned}$$

By (10.3), e and E are of the form

$$\begin{aligned}
e = {} & \alpha(x_{123} + x_{231} + x_{312} - x_{321} - x_{213} - x_{132}) \\
& + 2\alpha(x_{112} - x_{211}) - 2\alpha(x_{113} + x_{311}) + 2\alpha\, x_{212} \\
& + \beta\, x_{111} + \delta(x_{122} + x_{212} + x_{221}) \\
& + 2\delta(x_{112} - x_{211}) - 2\delta(x_{113} + x_{131} + x_{311}) \\
E = {} & \alpha'(x^{321} + x^{213} + x^{132} - x^{123} - x^{231} - x^{312}) \\
& + 2\alpha'(x^{332} - x^{233}) - 2\alpha'(x^{331} + x^{133}) + 2\alpha' x^{232} \\
& + \beta' x^{333} + \delta'(x^{322} + x^{232} + x^{223}) \\
& + 2\delta'(x^{332} - x^{233}) - 2\delta'(x^{331} + x^{313} + x^{133})
\end{aligned}$$

Now (10.4) reads

$$2(\alpha\delta' + \alpha'\delta) + 9\delta\delta' = 0 \qquad \alpha\alpha' \neq 0$$

We normalize to $\alpha = \alpha' = 1$. Next, (3.2g) $-$ ${}^{\tau}$(3.2h) reads $(\delta' - 2\delta)(\delta - 2\delta') = 0$, so up to Dynkin flip, we may assume that $\delta' = 2\delta$. We now have two cases.

Case IV.a: If $\delta = 0$, then (3.2gh) are equivalent to $\beta + \beta' + 2 = 0$. Using $\mathrm{Stab}(Q) = \{\begin{pmatrix} \lambda & \mu & \nu \\ 0 & \lambda & \mu \\ 0 & 0 & \lambda \end{pmatrix}\}$, we may get $\beta = \beta' = -1$; this gives the solution

$$(\alpha, \beta, \delta) = (1, -1, 0) \qquad (\alpha', \beta', \delta') = (1, -1, 0)$$

Case IV.b: If $\delta = -\frac{1}{3}$, then (3.2gh) are equivalent to $\beta' = -\frac{8}{27}$. Using $\mathrm{Stab}(Q)$, we may get $\beta = 0$; this gives the solution

$$(\alpha, \beta, \delta) = (1, 0, -\frac{1}{3}) \qquad (\alpha', \beta', \delta') = (1, -\frac{8}{27}, -\frac{2}{3})$$

10.8. **Type IV'.** We take C, D, c, d as for type IV. Condition (10.3) prevents e, E from satisfying (10.4), so this type is impossible.

11. Further problems

For $G = \mathrm{SL}(3)$, some technical problems and some links with other literature should be worthwhile studying:

- Replace the case by case argument in the proof of proposition 5.3 by a more conceptual one, preferrably including the elliptic case.
- Classify the elliptic solutions more explicitely, i.e., study conditions (10.5) (which define a subvariety in $\mathbb{P}^2 \times \mathbb{P}^2$).
- View each quantum SL(3) with $\omega = 1$ as a formal 1-parameter deformation of SL(3), compute the Lie bialgebra structure on $\mathfrak{sl}(3)$ at its semi-classical limit (see, e.g., [6] for definitions) and compare with the classification of these structures given in [20]. (Obviously, a quantum SL(3) with $\omega \neq 1$ cannot

be viewed as such a deformation.) Note that a related converse problem—that of finding an R-matrix quantizing each of the Lie bialgebra structures on $\mathfrak{sl}(3)$—has recently been solved in [11].

- Determine which quantum SL(3)'s admit a compact form. (The standard quantum SL(2) has a compact form, but the Jordanian one does not [18].)
- Our definition of a BQD seems to be related to the spiders of type A_2 introduced in [17].
- The shape algebras $\mathcal{M}_{\mathcal{L}}$ and $\mathcal{N}_{\mathcal{L}}$ from section 5 being homogeneous quadratic algebras, try to use the methods of [3] to associate geometric data to such a pair of algebras, in order to get a better understanding of the classification of quantum SL(3)'s. Incidentally, we note that the matrix Q used in section 10 is the same as that used in [1] to classify regular algebras of dimension 3 (here, such algebras would arise as quantum analogues of the homogeneous coordinate ring of $\mathrm{SL}(3)/P$, P a maximal parabolic subgroup).

Another problem is of course to extend the methods used here to study quantum G's for some other reductive group G. However, a classification is probably out of reach, even for $G = \mathrm{SL}(4)$.

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